LINEAR INDEPENDENCE OF GAMMA VALUES IN POSITIVE CHARACTERISTIC

W. DALE BROWNAWELL AND MATTHEW A. PAPANIKOLAS

ABSTRACT. We investigate the arithmetic nature of special values of Thakur's function field Gamma function at rational points. Our main result is that all linear dependence relations over the field of algebraic functions are consequences of the Anderson-Deligne-Thakur bracket relations.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Transcendence of Gamma values.** Let $\mathbb{F}_q$ be the field of q elements, where q is a power of p. Let $A := \mathbb{F}_q[\theta]$, $k := \mathbb{F}_q(\theta)$ for a variable θ. Let C_∞ be the completion of the algebraic closure of the completion $\mathbb{F}_q((1/\theta))$ with respect to the non-archimedean absolute value on k for which $|\theta| = q$. Let $A_+ := \{a \in A : a \text{ is monic}\}$ be the "positive integers" of A.

In this setting, D. Thakur defined a Gamma function

$$\Gamma(z) := \frac{1}{z} \prod_{n \in A_+} \left(1 + \frac{z}{n}\right)^{-1},$$

which is meromorphic on C_∞ with poles at the "negative" integers $-n \in -A_+$. One recognizes immediately a strong analogy with the classical Euler Gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.$$

Date: October 5, 1999.

1991 *Mathematics Subject Classification.* 11G09, 11J93, 11S80.

Key words and phrases. soliton t-modules, transcendence, linear independence, Gamma values, complex multiplication, quasi-periods.

The research of the first author was partially supported by an NSF grant and the second author by K. Ono's Sloan Research Fellowship.

Thakur's function shares other striking features with the Euler Gamma function, such as various natural functional equations, and rational (or infinite) values at the integers.

After isolated results by Thakur in [19], S. Sinha established the first transcendence results for general classes of values of the Gamma function.

Theorem 1.1.1 (Sinha [15], [16, §6.2]). *Let $a,\ f \in A_+$, $b \in A$, with $(a,f) = 1$, $\deg a < \deg f$. Then $\Gamma(\frac{a}{f} + b)$ is transcendental over k.*

Our goal is to extend this result, treating several values at once and evaluating Γ at more general arguments. There are however some natural dependencies.

1.2. **Dependence of Gamma Values.** In [19] Thakur established algebraic relations on Gamma values in this setting which are analogues of well-known relations of Anderson and Deligne for the classical Gamma function (see [1] or [23]). These relations express certain ratios of Gamma values at rational arguments as algebraic multiples of powers of the Carlitz period,

$$\widetilde{\pi} = \theta \sqrt[q-1]{-\theta} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \quad \in \mathbb{F}_q(\!(1/\theta)\!) \cdot \sqrt[q-1]{-\theta},$$

where $\sqrt[q-1]{-\theta}$ is some fixed $(q-1)$-st root of $-\theta$. The quantity $\widetilde{\pi}$ is the fundamental period of the Carlitz module (see D. Goss [11]).

The Anderson-Deligne-Thakur relations on Gamma values, derived from the formalism of *brackets* (see [17] or [19]), may be expressed as follows. Fix $f \in A_+$ and define

$$\mathcal{N}_f := \{a \in A : a \not\equiv 0 \bmod f\}, \qquad \mathcal{U}_f := \{u \in A : (u,f) = 1\},$$
$$\mathcal{M}_f := \{a \in \mathcal{N}_f : a \equiv m \bmod f, m \text{ monic}, \deg m < \deg f\}.$$

Then for any $u \in \mathcal{U}_f$, multiplication by u permutes the set $\mathcal{N}_f$. This action of $\mathcal{U}_f$ on $\mathcal{N}_f$ induces an action of $\mathcal{U}_f$ on $\oplus_{\mathcal{N}_f}\mathbb{Z}$, which we denote $u * \mathbf{m}$.

Our restatement of the bracket relations involves the sum of the coordinates of $\mathbf{m} := (m_a)_{a\in\mathcal{N}_f} \in \oplus_{\mathcal{N}_f}\mathbb{Z}$ having monic indices modulo f:

$$\Sigma_+(\mathbf{m}) := \sum_{a\in\mathcal{M}_f} m_a.$$

The following theorem gives the known (and, it is believed, all) $\overline{k}$-algebraic relations on Gamma values at rational points.

Definition. For $x,\ y \in C_\infty$, we write $x \sim y$ if $x/y \in \overline{k}$.

Theorem 1.2.1 (Thakur [19, §7.8]). *Let $f \in A_+$. Suppose that $\mathbf{m} = (m_a) \in \oplus_{\mathcal{N}_f}\mathbb{Z}$ and that $\Sigma_+(\mathbf{m}) = \Sigma_+(u * \mathbf{m})$, for any (and thus every) choice of representatives u of elements from $(A/f)^\times$. Then*

$$\prod_{a\in\mathcal{N}_f} \Gamma\left(\frac{a}{f}\right)^{m_a} \sim \quad \widetilde{\pi}^{\Sigma_+(\mathbf{m})}.$$

An equivalent version of this theorem was first formulated by G. Anderson [19, §7] and then proven by Thakur. Moreover Sinha obtained information on the quotient of the two sides in the above theorem [17, Theorem 2.2.4 and Remark 3.3.6].

The Gamma function satisfies several functional equations, directly analogous to the functional equations of the classical Gamma function [19]. Thakur's theorem includes as special cases the algebraic relations on special values determined by the functional equations mentioned above [19]. For example, for all $r \in k \smallsetminus A$, $a \in A$, g non-zero in A_+, $\deg g = d$,

$$\Gamma(r+a) \sim \Gamma(r)\,, \tag{1.1}$$

$$\prod_{\theta \in \mathbb{F}_q^\times} \Gamma(\theta r) \sim \widetilde{\pi}, \tag{1.2}$$

$$\prod_{\alpha \in A/(g)} \Gamma\Big(\frac{r+\alpha}{g}\Big) \sim \widetilde{\pi}^{\frac{q^d-1}{q-1}} \Gamma(r)\,. \tag{1.3}$$

However, when $q > 2$, the algebraic relations on special values induced by the functional equations are strictly subsumed by the bracket relations (see [15, Theorem VII.1]).

Definition. We adopt the notation $\Gamma(a/f) \approx \Gamma(b/f)$ to indicate that the relation $\Gamma(a/f) \sim \Gamma(b/f)$ follows from the bracket relations of Theorem 1.2.1 with $m_a = 1, m_b = -1$, and $m_c = 0$ for all the other entries in $\mathbf{m}$.

1.3. **Main Theorem.** The main result of this paper is that all the $\overline{k}$-linear relations on $\widetilde{\pi}$ and the finite values $\Gamma(r)$, for $r \in k \smallsetminus A$ are consequences of the bracket relations.

Theorem 1.3.1. *Let $r_1, \dots, r_n \in k \smallsetminus A$. Then the values 1, $\widetilde{\pi}$, $\Gamma(r_1), \dots, \Gamma(r_n)$ are $\overline{k}$-linearly independent unless for some $1 \leq i < j \leq n$, $\Gamma(r_i) \approx \Gamma(r_j)$.*

There are a number of noteworthy corollaries. In particular, we extend Sinha's result to all possible rational arguments r.

Corollary 1.3.2. *For each non-zero $r \in k \smallsetminus A$, $\Gamma(r)$ is transcendental.*

Another easily stated consequence is the following:

Corollary 1.3.3. *Let $r_1, \dots, r_n \in k$ be distinct with prime power denominators and with the numerator of each r_i having degree less than that of the denominator of r_i. Then the values 1, $\widetilde{\pi}$, $\Gamma(r_1), \dots, \Gamma(r_n)$ are $\overline{k}$-linearly independent.*

This corollary can be extended in the following manner:

Corollary 1.3.4. *Let $f = \prod f_i^{e_i}$ be the decomposition of f in A in terms of distinct irreducible factors f_i. If no f_i divides any $f_j - 1$, then the numbers*

$$1,\, \widetilde{\pi},\, \Gamma(a/f), \quad a \in A,\ 0 \leq \deg a < \deg f,$$

are $\overline{k}$-linearly independent.

Note that we specifically allow $(a, f) \neq 1$.

The formulation of our main result clearly resembles that of Satz 4 of [23], where the analogous result is proven for values of the classical Beta function at rational points. There the analogue of the bracket relations of Anderson-Deligne-Thakur are the relations on the Beta values which arise from the classical relations for values of the classical Gamma function, whereas, as noted above, certain relations on values do *not* follow from functional relations for $\Gamma(z)$. The analogue of our result for values of the classical Gamma function itself is still unknown except for very special cases due to Th. Schneider and G.V. Chudnovsky.

Seen on a large enough scale, the proofs here and in [23] also run somewhat parallel, based as they are on G. Wüstholz's Theorem of the Subgroup and J. Yu's Theorem of the Sub-t-module, respectively. Luckily, as stated above, the theory of bracket relations even provides the analogue of the Deligne-Koblitz-Ogus characterization for the algebraicity of the product of values of the (normalized) classical Gamma function at rational points predicted by the classical relations.

However some of the crucial tools of [23] were not available for application to t-modules. In particular, we lacked analogues of the following:

(a) Poincaré's complete decomposability (up to isogeny) of abelian varieties into products of simple ones.

(b) The Shimura-Tanayama criterion for the explicit decomposition of the Jacobian of the Fermat curve into simple varieties of CM-type; indeed the very notion of a Jacobian is missing from our context.

(c) An interpretation of all Beta values at rational points as abelian integrals. (Sinha provides a full analogue only when $q = 2$.)

In the next subsection, we describe in general terms how we proceed in this paper.

1.4. **Outline of the Paper.** In order to apply the transcendence machine embodied in the Theorem of the Sub-t-module, we obviously require appropriate t-modules.

In Section 2 we review some basic definitions. We then introduce t-modules of CM-type and show that, up to isogeny, they are always powers of simple t-modules of CM-type. We also give natural criteria, in terms of the underlying CM structure, for determining the simplicity of t-modules of CM-type and for determining whether the simple t-modules underlying two given t-modules of CM-type are isogenous. These criteria, although of a vastly different nature, play a role in our transcendence considerations somewhat analogous to the above mentioned Shimura-Taniyama criterion.

In Section 3 we define biderivations for arbitrary t-modules E and construct their associated quasi-periodic extensions Q. We show that, when the biderivations represent linearly independent classes modulo the inner biderivations, the resulting quasi-periodic extension is minimal in a precise sense. The exponential function of Q comprises the components of the exponential function of E as well as the quasi-periodic functions associated to the representative biderivations (plus new variables corresponding to $\mathbb{G}_a^j$).

We note that an isogeny $T : E_1 \to E_2$ between t-modules lifts to an isogeny $T_* : Q_1 \to Q_2$ between the minimal quasi-periodic extensions of maximal dimension. As a result, in the case of quasi-periodic extensions of t-modules of CM-type, we develop a serviceable version of Poincaré's theorem.

Section 4 is the heart of the paper. The main work of [15, 16] involves the construction and investigation of a t-module E_f whose periods have coordinates which are algebraic multiples of the quantities

$$\Gamma(a/f), \quad a \in A_+,\ \deg(a) < \deg(f),\ (a, f) = 1, \tag{1.4}$$

where $f \in A_+$ is fixed. Sinha does this by generalizing Drinfeld's shtuka version of Drinfeld modules to higher dimensional t-modules. Starting with Anderson's two-variable "soliton" function, which generalizes some one-variable results of R. Coleman, Sinha creates a module E_f which is of CM-type.

We observe that the bracket relations of Section 1.2 reflect precisely the CM-structure of E_f. At this point we could deduce that the set of $\Gamma(a/f)$, a, $f \in A_+$, $a \in \mathcal{U}_f$, is $\overline{k}$-linearly independent if and only if no pair of these Gamma values is $\overline{k}$-linearly dependent.

To provide a setting in which the remaining Gamma values occur, we consider certain quasi-periodic t-module extensions Q_f of E_f by $\mathbb{G}_a^j$:

$$0 \to \mathbb{G}_a^j \to Q_f \to E_f \to 0. \tag{1.5}$$

We compute the periods of these Q_f and see that their components are algebraic multiples of all the values

$$\Gamma(a/f), \quad \deg(a) < \deg(f),\ (a, f) = 1,$$

where a, $f \in A$. This removes the restriction that a and f be monic.

In Section 5 we prove independence results about periods of quasi-periodic extensions of t-modules of CM-type. Basic properties of minimal extensions enable us to apply Yu's theorem to a product $Q_{f_1} \times \cdots \times Q_{f_m}$ of quasi-periodic t-modules Q_{f_j} of the sort in (1.5) . Each such quasi-periodic extension Q_f is minimal and is in fact, up to isogeny, the power of a minimal extension of a t-module of CM-type. Therefore, the question of linear independence of the coordinates of a period of $Q_{f_1} \times \cdots \times Q_{f_m}$ is reduced to the question of the corresponding periods of $E_{f_1} \times \cdots \times E_{f_m}$. Theorem 1.3.1 then follows directly from the isogeny criterion and Yu's theorem.

In Section 6 we present a few examples.

Acknowledgements. The authors have greatly benefited from the encouragement of various colleagues. In particular, we would like to thank Greg Anderson, David Goss, Marius van der Put, and Dinesh Thakur who have provided advice and invaluable assistance at various turns.

2. t-modules and t-motives of CM-Type

2.1. **General Definitions.** We review some basic facts about t-modules and t-motives in order to establish notation. Complete accounts of the material sketched in this section are contained in the standard references [2] and Chapter 5 of [11]. Continuing with the notation of Section 1, let $k \subset L \subset C_\infty$ with L algebraically closed.

Remark. We will also need another copy of the pair A, k which we keep separate, as they will be associated with "operators" rather than the scalars of C_∞. We denote the new variable by t and the polynomial ring and the fields by the Euler fonts: $\mathsf{A} := \mathbb{F}_q[t]$, $\mathsf{k} := \mathbb{F}_q(t)$. We let $\iota : \mathsf{k} \to k$ be the isomorphism sending $t \mapsto \theta$. In general, the fonts, A, B, K will be reserved for rings of operators. We will not maintain this font distinction for elements, and so whether we consider $f \in A$ or $f \in \mathsf{A}$ will depend on the context. Nevertheless, we will persist in the distinction between $\theta \in A$ and $t \in \mathsf{A}$.

Let τ denote the q-th power Frobenius map: $x \mapsto x^q$, $q = p^r$.

Let E be an algebraic group defined over L isomorphic to $\mathbb{G}_a^d$. We take $\mathrm{Lie}(E)$ for its tangent space at the origin and note that, after choosing a basis for this isomorphism, $\mathrm{Lie}(E) \simeq L^d$. Similarly, if $\mathrm{End}_L^q(E)$ is the ring of $\mathbb{F}_q$-linear endomorphisms of E as an algebraic group over L, then selecting a basis induces an isomorphism

$$\mathrm{End}_L^q(E) \simeq \mathrm{Mat}_{d\times d}(L\{\tau\}) \simeq \mathrm{Mat}_{d\times d}(L)\{\tau\},$$

where for any $\mathbb{F}_q$-algebra R we take $R\{\tau\}$ for the non-commutative ring of *twisted polynomials in* τ.

2.1.1. *t-modules and t-motives.* A *t-module over* L is an algebraic group E, isomorphic to $\mathbb{G}_a^d$ over L, for which there is an $\mathbb{F}_q$-linear homomorphism

$$\Phi : \mathsf{A} \to \mathrm{End}_L^q(E)$$

and an $l > 0$ such that the endomorphism $\Phi(t) - \theta\tau^0$ satisfies

$$(\Phi(t) - \theta\tau^0)^\ell \, \mathrm{Lie}(E) = \{0\}.$$

We refer to d as the *dimension* of E. A sub-t-module H of E is a connected sub-algebraic group of E which is also invariant under the action of $\Phi(t)$. A t-module is said to be *simple* if it does not contain any proper sub-t-modules. A *morphism* $\Theta : E_1 \to E_2$ of t-modules over L is a morphism of algebraic groups over L which commutes with the actions of t on E_1 and E_2. We let $\mathrm{Hom}(E_1, E_2)$ denote the group of t-module morphisms $E_1 \to E_2$ and also take $\mathrm{End}(E) := \mathrm{Hom}(E_1, E_1)$. A morphism Θ will be called an *isogeny* if the dimensions of E_1 and E_2 are the same and if Θ has a finite kernel as a map of algebraic groups. We will write $E_1 \sim E_2$ to denote that E_1 is isogenous to E_2; furthermore, isogeny is an equivalence relation (see [25, Lemma 1.1]).

By choosing an isomorphism $E \simeq \mathbb{G}_a^d$, we obtain an $\mathbb{F}_q$-linear homomorphism

$$\Phi : \mathsf{A} \to \mathrm{Mat}_{d\times d}(L)\{\tau\}$$

which provides the action of A on E. If Φ is defined by

$$\Phi(t) = M_0\tau^0 + M_1\tau + \cdots + M_n\tau^n,$$

then the induced action on $\mathrm{Lie}(E)$ is given by $d\Phi(t) := M_0$. The definition of a t-module dictates that $d\Phi(t) = (\theta I_d + N)$ for the $d \times d$ identity matrix I_d and a nilpotent matrix N. To signify that we have chosen a system of coordinates for E we will write $E = (\Phi, \mathbb{G}_a^d)$. Thus if $E_1 = (\Phi_1, \mathbb{G}_a^{d_1})$, $E_2 = (\Phi_2, \mathbb{G}_a^{d_2})$ are two t-modules, a morphism $\Theta : E_1 \to E_2$ is a matrix of twisted polynomials, $\Theta \in \mathrm{Mat}_{d_2\times d_1}(C_\infty\{\tau\})$, satisfying

$$\Theta\Phi_1(t) = \Phi_2(t)\Theta.$$

The dual notion of a t-module is that of a t-motive. Set $L[t,\tau] := L\{\tau\}[t]$, the ring of commuting polynomials in the variable t over the non-commutative ring $L\{\tau\}$. Then a *t-motive* M is a left $L[t,\tau]$-module which is free and finitely generated as an $L\{\tau\}$-module and for which

$$(t-\theta)^\ell (M/\tau M) = \{0\},$$

for some $\ell > 0$. Morphisms of t-motives are morphisms of left $L[t,\tau]$-modules.

To every t-module $E = (\Phi, \mathbb{G}_a^d)$ over L, there corresponds a unique t-motive over L:

$$M := M(E) := \mathrm{Hom}_L^q(E, \mathbb{G}_a),$$

where $\mathrm{Hom}_L^q(A, B)$ denotes the L-module of $\mathbb{F}_q$-linear morphisms of algebraic groups A, B over L. In this setting, the action of ft^i, $f \in L\{\tau\}$ on $m \in M$ is

$$(ft^i, m) \mapsto f \circ m \circ \Phi(t^i).$$

Projections on the d coordinates of $E \simeq \mathbb{G}_a^d$ form an $L\{\tau\}$-basis for M; $d = \mathrm{rank}_{L\{\tau\}} M$; and we can take $\ell \leq d$.

It is a fundamental theorem of Anderson that the functor $E \mapsto M(E)$ gives an anti-equivalence from the category of t-modules over L to the category of t-motives over L. Given a t-motive M together with and $L\{\tau\}$-basis $m_1, \ldots, m_d$ for M, we can express the t-action with respect to this basis:

$$t \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \Phi(t) \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix}, \tag{2.1}$$

where $\Phi(t) \in \mathrm{Mat}_{d\times d}(L\{\tau\})$.

We create a t-module $E = (\Phi, \mathbb{G}_a^d)$ in the following way:

Elements m of M correspond uniquely to $\boldsymbol{U} = (U_1, \ldots, U_d) \in \mathrm{Mat}_{1\times d}(L\{\tau\})$ via

$$m = (U_1, \ldots, U_d) \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix}.$$

According to the commutativity of t with elements of $L\{\tau\}$,

$$t \cdot \boldsymbol{U} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \boldsymbol{U} \cdot t \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \boldsymbol{U}\Phi(t) \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix}.$$

In other words, we can take the action of t on $\mathrm{Mat}_{1\times d}(L\{\tau\})$ to be

$$t \cdot \boldsymbol{U} = \boldsymbol{U}\Phi(t).$$

If we now define $E := (\Phi, \mathbb{G}_a^d)$, then the action of t on M with respect to the basis $m_1, \ldots, m_d$ is that of $M(E)$. In particular, when the t-motive M is defined over L, so is its corresponding t-module E. This remark will be essential for our transcendence considerations below.

2.1.2. *Exponential Function and Uniformization.* Given a t-module $E = (\Phi, \mathbb{G}_a^d)$ of defined over L, there is a unique associated *exponential function*

$$\mathrm{Exp} = \mathrm{Exp}_E : \mathrm{Lie}(E) \to E(C_\infty),$$

which satisfies

(a) $\mathrm{Exp}(d\Phi(t)\mathbf{z}) = \Phi(t)\,\mathrm{Exp}(\mathbf{z})$.

(b) $\partial\,\mathrm{Exp}(\mathbf{z}) = I_d$.

Here $\partial\,\mathrm{Exp}(\mathbf{z})$ denotes the matrix of coefficients of linear terms in $\mathrm{Exp}(\mathbf{z})$. The function Exp is an entire $\mathbb{F}_q$-linear function $\mathrm{Exp} : C_\infty^d \to C_\infty^d$ with coefficients from L. (A power series is $\mathbb{F}_q$-*linear* if it is a sum of power series in single variables in which each exponent is a power of q.)

Elements of $\Lambda := \mathrm{Ker}\,\mathrm{Exp}$ are called *periods* of E. A major effort of the paper is devoted to the definition of certain t-modules whose periods involve the special Gamma values.

2.1.3. *Abelian t-modules and Uniformization.* The t-module E is called *uniformizable* if the image of Exp is all of $\mathbb{G}_a^d(C_\infty)$. Conditions for surjectivity of the exponential function is a quite fascinating topic; we hope to give new criteria for uniformizability in a future note.

When a t-motive M is finitely generated over $L[t]$, it and the corresponding t-module E are said to be *abelian*. In fact M is abelian precisely when it is free of finite rank over $L[t]$; we call this rank the *rank* of M and E and denote it by $r(M) = r(E)$. By a theorem of Anderson [2, Theorem 4], [11, Theorem 5.9.14], the uniformizability of an abelian t-module E is equivalent to the condition

$$\operatorname{rank}_{\mathsf{A}}(\operatorname{Ker}\operatorname{Exp}_E) = \operatorname{rank}_{C_\infty[t]} M(E). \tag{2.2}$$

Furthermore, if E is uniformizable, then a theorem of Anderson [2, Cor. 3.3.6] shows that Λ spans $\operatorname{Lie}(E)$ over C_∞.

Remark 2.1.1. It should be noted that if E is abelian and uniformizable, then every sub-t-module H is also abelian and uniformizable. Indeed, it is easy to see that E is abelian if and only if both H and the quotient t-module E/H are abelian as well. By Yu [24, Prop. 5.3], we know that H must be uniformizable if E is.

Let $E = (\mathbb{G}_a^d, \Phi)$ be an abelian t-module. If T is a finite subgroup of $E(C_\infty)$, which is invariant under the action of $\Phi(t)$, then the quotient E/T (as algebraic groups) is naturally given the structure of a t-module (see [11, §5.6]). The map of t-modules $E \to E/T$ is a surjective map of algebraic groups. If E is uniformizable, it then follows that E/T is also uniformizable.

Lemma 2.1.2. *Let $E = (\Phi, \mathbb{G}_a^d)$ be a uniformizable abelian t-module with period lattice Λ. Let $\Lambda' \subset \operatorname{Lie}(E)$ be an A-lattice in which Λ has finite index. Then Λ' is the period lattice of a uniformizable abelian t-module $E' = (\Phi', \mathbb{G}_a^d)$, and there is a natural isogeny $E \to E'$.*

Proof. Let $T \subset E(C_\infty)$ be the image of Λ' under Exp_E. Then T is finite and invariant under $\Phi(t)$. Let $E' = E/T$. We can identify the tangent spaces of E and E', and by the functoriality of exponential functions, the exponential function of E' is the map

$$\operatorname{Exp}_{E'} : \operatorname{Lie}(E) \xrightarrow{\operatorname{Exp}_E} E(C_\infty) \to E'(C_\infty),$$

which is surjective and has kernel Λ'. □

2.2. **Hilbert-Blumenthal-Drinfeld Modules.** Let K_+/k be a finite separable extension with $[\mathsf{K}_+ : \mathsf{k}] = d$ such that the place ∞ of k is totally split in K_+, and take B_+ for the integral closure of A in K_+. Let $\sigma_1, \dots, \sigma_d$ be the embeddings of $\mathsf{K}_+ \hookrightarrow C_\infty$. Define the *conjugate action* $\boldsymbol{\sigma} := \sigma_1 \oplus \cdots \oplus \sigma_d$ of K_+ on C_∞^d in the following manner: For $b \in \mathsf{K}_+$ and $\mathbf{z} := (z_1, \dots, z_d) \in C_\infty^d$,

$$\boldsymbol{\sigma}(b) : (z_i) \mapsto \boldsymbol{\sigma}(b)(\mathbf{z}) := (\sigma_i(b) z_i). \tag{2.3}$$

Suppose now that $E = (\Phi, \mathbb{G}_a^d)$ is a uniformizable abelian t-module and that Φ extends to a map of A-algebras

$$\Phi : \mathsf{B}_+ \to \operatorname{End}(E)$$

in such a way that the action of $d\Phi$ on $\operatorname{Lie}(E)$ is given by

$$d\Phi(b) = \boldsymbol{\sigma}(b), \ \forall b \in \mathsf{B}_+.$$

Then E is called a *Hilbert-Blumenthal-Drinfeld module* (H-B-D module) with multiplications by B_+. By definition the period lattice Λ of E is invariant under $\boldsymbol{\sigma}(\mathsf{B}_+)$. Working with Hilbert-Blumenthal-Drinfeld modules is greatly facilitated by the following equivalence.

Theorem 2.2.1 (Anderson [2, Theorem 7]). *The following two categories are equivalent.*

(a) *Objects: H-B-D modules with multiplications by* B_+.
Morphisms: t-module homomorphisms which are B_+*-equivariant.*

(b) *Objects: Lattices contained in* C_∞^d *invariant under* $\boldsymbol{\sigma}(\mathsf{B}_+)$.
Morphisms: C_∞*-linear maps on* C_∞^d *which stabilize the lattice and commute with* $\boldsymbol{\sigma}(\mathsf{B}_+)$.

2.3. **t-modules of CM-type.** One particular sort of H-B-D module, called *t-modules of CM-type*, are of prime importance in this paper. We continue with the notation of the previous section. Let K be a finite extension of K_+ which is totally ramified at each infinite place of K_+. In this situation, we say that K_+ is the *maximal real subfield of* K. For ease of exposition we will assume that K_+/k and K/k are both Galois extensions, although this is not strictly necessary.

Choose extensions of $\sigma_1, \ldots, \sigma_d$ to embeddings $\mathsf{K} \hookrightarrow C_\infty$ which we also denote by $\sigma_1, \ldots, \sigma_d$. Letting

$$\mathcal{S} := \{\sigma_1, \ldots, \sigma_d\} \subset \mathrm{Gal}(\mathsf{K}/\mathsf{k}) \tag{2.4}$$

be the set of these extensions, we denote the extension of the conjugate action $\boldsymbol{\sigma}$ to K by $\boldsymbol{\sigma}_{\mathcal{S}}$.

Let Λ be a discrete A-submodule of C_∞^d of rank $[\mathsf{K} : \mathsf{k}]$ which is invariant under the action via $\boldsymbol{\sigma}_{\mathcal{S}}$ of some order of K. We denote by B the maximal such order. As a B-module, Λ is isomorphic to an ideal of B. We think of Λ as having *real multiplications* by the conjugate action $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B}_+ \cap \mathsf{B})$ and *complex multiplications* by the conjugate action $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B})$, *CM by* B for short.

By Anderson's Theorem 2.2.1, Λ contains a sublattice of finite index which is the period lattice of a H-B-D module. By Lemma 2.1.2, Λ is then itself the period lattice of a uniformizable abelian t-module $E = (\Phi, \mathbb{G}_a^d)$. Because $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B})\Lambda \subset \Lambda$ and the action of B via $\boldsymbol{\sigma}_{\mathcal{S}}$ commutes with the action of $\mathsf{B}_+ \cap \mathsf{B}$ via $\boldsymbol{\sigma}$, Anderson's theorem also gives an extension of Φ to all of B such that

$$d\Phi(b) = \boldsymbol{\sigma}_{\mathcal{S}}(b)$$

for all $b \in \mathsf{B}$. In other words, we have an injection of A-algebras

$$\Phi : \mathsf{B} \hookrightarrow \mathrm{End}(E), \tag{2.5}$$

which extends the t-module homomorphism Φ on A. We call K the *CM-field* of E and say E has *CM-type* $(\mathsf{K}, \mathcal{S})$.

Remark. In the case that Λ is isomorphic to a principal ideal of B, fixing a generator $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)^{tr} \in C_\infty^d$ allows us to identify B with $\Lambda \subset C_\infty^d$ via $\boldsymbol{\sigma}$:

$$\begin{aligned} \mathsf{B} &\xrightarrow{\sim} \Lambda \subset C_\infty^d \\ b &\mapsto \begin{pmatrix} \sigma_1(b)\lambda_1 \\ \vdots \\ \sigma_d(b)\lambda_d \end{pmatrix} =: \boldsymbol{\sigma}_{\mathcal{S}}(b)(\boldsymbol{\lambda}). \end{aligned} \tag{2.6}$$

Yu [24, Lemma 6.2] shows that all the coordinates λ_i of $\boldsymbol{\lambda}$ are non-zero.

Remark 2.3.1. If two t-modules E_1 and E_2 have the same CM-type $(\mathsf{K}, \mathcal{S})$, then they are necessarily isogenous. Indeed if E_1 has CM by B_1 and E_2 has CM by B_2, then let $\mathsf{B} = \mathsf{B}_1 \cap \mathsf{B}_2$. The period lattices Λ_1 and Λ_2 both contain sublattices which, as in (2.6) are isomorphic to B via $\boldsymbol{\sigma}_{\mathcal{S}}$. Thus both E_1 and E_2 are isogenous to t-modules which have period lattices isomorphic to $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B})$. By Anderson's Theorem 2.2.1 and Lemma 2.1.2, we see that these t-modules are themselves isogenous, and therefore $E_1 \sim E_2$.

2.4. **Endomorphism Rings.** Throughout this section we will assume that $E = (\Phi, \mathbb{G}_a^d)$ is an abelian t-module with period lattice Λ. Let $\mathrm{End}^0(E) := \mathrm{End}(E) \otimes_{\mathsf{A}} \mathsf{k}$. Certainly Φ extends to a map $\Phi : \mathsf{k} \to \mathrm{End}^0(E)$. Our goal of this section is to determine the structures of $\mathrm{End}(E)$ and $\mathrm{End}^0(E)$.

Lemma 2.4.1. *Let E be simple. Then the ring $\mathrm{End}^0(E) := \mathrm{End}(E) \otimes_{\mathsf{A}} \mathsf{k}$ is a division algebra and k lies in its center.*

Proof. $\mathrm{End}(E)$ has no non-zero zero divisors because E is assumed to be simple. Now if $f \in \mathrm{End}(E)$, then the kernel of f is a sub-algebraic group invariant under the action of t. Its connected component will be a sub-t-module of E, and so the kernel of f must be finite. Thus Lemma 1.1 of [25] gives $a \in \mathsf{A}$ and $g \in \mathrm{End}(E)$ so that, (as elements of $\mathrm{End}(E)$),

$$fg = \Phi(a).$$

Therefore $\mathrm{End}^0(E)$ is a division algebra, and its center clearly contains k. □

Proposition 2.4.2. *Suppose E is isogenous to a product $E_1^{n_1} \times \cdots \times E_k^{n_k}$ of powers of pair-wise non-isogenous, simple abelian t-modules. Then*

$$\mathrm{End}^0(E) \simeq \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i \times n_i}(\mathrm{End}^0(E_i)).$$

Proof. Replace the word "isomorphism" in the statement and proof of Proposition 1.2 of [13, Chapter XVII] by "isogeny". □

For the rest of this section we assume that E is simple, let $s := \mathrm{rank}_{\mathsf{A}} \Lambda$ and assume $s > 0$. Unless E is uniformizable, this rank s differs from the rank r of E as an abelian t-module.

Because E is simple, the image of each non-zero element $f \in \mathrm{End}(E)$ is all of E. Therefore the kernel of such an f is finite, and f must take Λ to a sublattice which has finite index in Λ. Hence there is an injection of A-algebras $\mathrm{End}(E) \hookrightarrow \mathrm{Mat}_{s\times s}(\mathsf{A})$ such that the image of every non-zero endomorphism is invertible in $\mathrm{Mat}_{s\times s}(\mathsf{k})$. This map extends uniquely to a map of k-algebras

$$\mathrm{End}^0(E) \hookrightarrow \mathrm{Mat}_{s\times s}(\mathsf{k}), \tag{2.7}$$

which is called the *rational representation* of $\mathrm{End}^0(E)$. We see right away that $\mathrm{End}(E)$ is a free A-module of rank over A at most s^2 and that $\dim_{\mathsf{k}} \mathrm{End}^0(E) \leq s^2$.

Proposition 2.4.3. *Let E be a simple abelian t-module with period lattice of rank s over A. Let K_0 be the center of $\mathrm{End}^0(E)$; $g = [\mathsf{K}_0 : \mathsf{k}]$; and $h^2 = [\mathrm{End}^0(E) : \mathsf{K}_0]$. Then $gh^2 \mid s$.*

Proof. As the rational representation of (2.7) is a faithful representation of the division algebra $\mathrm{End}^0(E)$, it follows that $[\mathrm{End}^0(E) : \mathsf{k}]$ must divide s (see [13, Prop. XVII.4.7]). ☐

Corollary 2.4.4. *Let $E = (\Phi, \mathbb{G}_a^d)$ be a simple Hilbert-Blumenthal-Drinfeld module of CM-type $(\mathsf{K}, \mathcal{S})$ with complex multiplications by B. Then complex multiplication gives an isomorphism $\Phi : \mathsf{B} \xrightarrow{\sim} \mathrm{End}(E)$ which lifts to an isomorphism $\mathsf{K} \xrightarrow{\sim} \mathrm{End}^0(E)$.*

Proof. Let K_0, g, h, s be as in Proposition 2.4.3. Now $\Phi(\mathsf{K})$ is a subfield of $\mathrm{End}^0(E)$, and $[\Phi(\mathsf{K}) : \Phi(\mathsf{k})] = [\mathsf{K} : \mathsf{k}] = r$, and $r = s$, as E is uniformizable. Since the centralizer of $\Phi(K)$ contains K_0, we conclude that $\Phi(\mathsf{K})\mathsf{K}_0$ is a subfield of $\mathrm{End}^0(E)$. Because each maximal commutative subfield of $\mathrm{End}^0(E)$ has dimension h over K_0 (see [11, Cor. 4.11.15]), we conclude that $[\Phi(\mathsf{K})\mathsf{K}_0 : \Phi(\mathsf{k})] \leq gh$. By Proposition 2.4.3 we know $gh^2 \mid r = [\mathsf{K} : \mathsf{k}]$, and so

$$r = [\Phi(\mathsf{K}) : \Phi(\mathsf{k})] \leq [\Phi(\mathsf{K})\mathsf{K}_0 : \Phi(\mathsf{k})] \leq gh \leq gh^2 \leq r.$$

Hence $r = gh = gh^2$ and so $h = 1$; thus $\mathrm{End}^0(E) \simeq \mathsf{K}$. Now $\mathrm{End}(E)$ is isomorphic to an order in K, and since B is the largest order $\mathcal{O}$ for which $\boldsymbol{\sigma}_{\mathcal{S}}(\mathcal{O})$ leaves Λ invariant, we must have $\mathrm{End}(E) = \Phi(\mathsf{B})$. ☐

Remark. It is possible to prove Corollary 2.4.4 more directly. If $f \in \mathrm{End}(E)$, then df leaves Λ invariant. Because $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{K})\Lambda = \boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{k})\Lambda$, we see that df leaves the 1-dimensional K-vector space $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{K})\Lambda$ invariant. Any non-zero element $\boldsymbol{\lambda} \in \Lambda$ is a generator for this vector space, and so there exists a unique $b \in \mathsf{K}^\times$ such that

$$df(\boldsymbol{\lambda}) = \boldsymbol{\sigma}_{\mathcal{S}}(b)(\boldsymbol{\lambda}).$$

We know that $\boldsymbol{\sigma}(b) \in \mathrm{End}^0(E)$ and we want to show, among other things, that $b \in \mathsf{B}$. However we know that, for some denominator $a \in \mathsf{A}$, we have $ab \in \mathsf{B}$. For $g := \Phi(a)f - \Phi(ab) \in \mathrm{End}(E)$, the displayed line shows that dg has a non-trivial kernel ($dg\boldsymbol{\lambda} = 0$). Because E is simple, every non-zero endomorphism is an isogeny and so is an isomorphism on $\mathrm{Lie}(E)$. Thus $g = 0$, i.e. $\Phi(a)f = \Phi(ab)$, and thus $\Phi(b) = f \in \mathrm{End}(E)$. As $\boldsymbol{\sigma}_{\mathcal{S}}(b)\Lambda = d\Phi(b)\Lambda = df\Lambda \subset \Lambda$, we see that $b \in \mathsf{B}$.

2.5. **Sub-t-modules and Isogenies.** In this section, we investigate criteria for determining when two t-modules of CM-type E_1 and E_2 have a non-trivial t-module morphism between them or, what will be equivalent, when the two have isogenous sub-t-modules. We describe completely the sub-t-module structure of E and thus ascertain what sorts of t-module morphisms exist between t-modules of CM-type.

Let E be a t-module of CM-type as in Section 2.2, with multiplication rings B and B_+ and fraction fields K and K_+ as described there. Let $\mathsf{G} = \mathrm{Gal}(\mathsf{K}/\mathsf{k})$ and $\mathsf{G}_+ = \mathrm{Gal}(\mathsf{K}/\mathsf{K}_+)$, and take $\mathcal{S} := \{\sigma_1, \dots, \sigma_d\} \subset \mathsf{G}$ for the conjugate embeddings giving E the structure of an H-B-D module, i.e. E has CM-type $(\mathsf{K}, \mathcal{S})$. The following lemma is the basis for the remainder of our discussion.

Lemma 2.5.1. *Let $\mathsf{L} \subset \mathsf{K}$ be a subfield, and set $\mathsf{L}_+ := \mathsf{L} \cap \mathsf{K}_+$. Then the following are equivalent:*

(a) $[\mathsf{L} : \mathsf{L}_+] = [\mathsf{K} : \mathsf{K}_+]$*; and for all i, j, if $\sigma_i|_{\mathsf{L}_+} = \sigma_j|_{\mathsf{L}_+}$, then $\sigma_i|_{\mathsf{L}} = \sigma_j|_{\mathsf{L}}$.*

(b) *$\mathcal{S}$ is the union of certain left cosets of* $\mathrm{Gal}(\mathsf{K}/\mathsf{L})$.

Proof. The proof is based on two straightforward remarks: (1) $\mathsf{K}_+\mathsf{L}/\mathsf{L}$ is a Galois extension with $\mathrm{Gal}(\mathsf{K}_+\mathsf{L}/\mathsf{L}_+) \simeq \mathrm{Gal}(\mathsf{K}_+/\mathsf{L}_+)$ (cf. [13, VI, Theorem 1.12]). (2) As $\mathcal{S}$ consists of the extensions to K of the distinct elements of $\mathrm{Gal}(\mathsf{K}_+/\mathsf{k})$, precisely $[\mathsf{K}_+ : \mathsf{L}_+]$ distinct elements of $\mathcal{S}$ restrict to each of the $[\mathsf{L}_+ : \mathsf{k}]$ embeddings of L_+ into K_+.

(a) $\Rightarrow$ (b): By the first part of (a) and (1), $[\mathsf{K} : \mathsf{L}] = [\mathsf{K}_+ : \mathsf{L}_+] = [\mathsf{K}_+\mathsf{L} : \mathsf{L}]$, so $\mathsf{K}_+\mathsf{L} = \mathsf{K}$.

By the second part of (a), elements of distinct cosets of $\mathrm{Gal}(\mathsf{K}/\mathsf{L})$ restrict to distinct embeddings of L_+. Thus according to (2), the intersection of $\mathcal{S}$ with a coset of $|\mathrm{Gal}(\mathsf{K}/\mathsf{L})|$ must contain 0 or $[\mathsf{K}_+ : \mathsf{L}_+] = [\mathsf{K} : \mathsf{L}] = |\mathrm{Gal}(\mathsf{K}/\mathsf{L})|$ elements. In other words, the intersection is either empty or the whole coset. This proves (b).

(b) $\Rightarrow$ (a): Since elements of a coset of $\mathrm{Gal}(\mathsf{K}/\mathsf{L})$ restrict to the same embedding of L_+ into K_+, (2) shows that $\mathcal{S}$ contains at least $[\mathsf{L}_+ : \mathsf{k}]$ such cosets, and

$$[\mathsf{K} : \mathsf{L}][\mathsf{L}_+ : \mathsf{k}] \leq |\mathcal{S}| = [\mathsf{K}_+ : \mathsf{k}] = [\mathsf{K}_+ : \mathsf{L}_+][\mathsf{L}_+ : \mathsf{k}].$$

Since, by (1), $[\mathsf{K}_+ : \mathsf{L}_+] = [\mathsf{K}_+\mathsf{L} : \mathsf{L}]$, we find $[\mathsf{K} : \mathsf{L}] \leq [\mathsf{K}_+\mathsf{L} : \mathsf{L}]$. Thus we must have that $\mathsf{K}_+\mathsf{L} = \mathsf{K}$. Therefore $[\mathsf{K} : \mathsf{L}] = [\mathsf{K}_+ : \mathsf{L}_+]$, and it follows that $[\mathsf{L} : \mathsf{L}_+] = [\mathsf{K} : \mathsf{K}_+]$, which is the first part of (a).

Now we know that each coset contains $[\mathsf{K} : \mathsf{L}] = [\mathsf{K}_+ : \mathsf{L}_+]$ elements, each of which restricts to the same embedding of L_+ into K_+. By (2) then, elements from distinct cosets of $\mathrm{Gal}(\mathsf{K}/\mathsf{L})$ in $\mathcal{S}$ must restrict to distinct embeddings of L_+ into K_+. This is the second part of (a). □

The following proposition provides the sub-t-module structure of E and determines when E is simple. It is similar to a well-known theorem about abelian varieties of CM-type (see Mumford [14, §19]).

Theorem 2.5.2. *Let E be a t-module of CM-type* $(\mathsf{K}, \mathcal{S}) = (\mathsf{K}, \{\sigma_1, \dots, \sigma_d\})$.

(a) *E is isogenous to a power of a simple sub-t-module.*

(b) *E is itself simple if and only if no proper subfield $\mathsf{L} \subset \mathsf{K}$ satisfies either of the equivalent criteria of Lemma 2.5.1.*

Remark. The simple sub-t-module provided by this proposition is itself a t-module of CM-type. Indeed the conditions of Lemma 2.5.1 dictate that it has CM-type $(\mathsf{L}, \mathcal{S}|_{\mathsf{L}})$.

Proof of Theorem 2.5.2. We first prove (a). Let $H \subset E$ be a simple sub-t-module. Let $\mathsf{C} \subset \mathsf{B}$ be the largest A-algebra for which $\Phi(\mathsf{C})$ leaves H invariant. Let L be the fraction field of C; thus C is an order of L. Let $d_0 = \dim H$; $m = [\mathsf{K} : \mathsf{L}]$; and $n = [\mathsf{L} : \mathsf{k}]$. For some $\beta_1, \dots, \beta_m \in \mathsf{B}$, the direct sum

$$\beta_1\mathsf{C} + \cdots + \beta_m\mathsf{C} \subset \mathsf{B},$$

is an A-submodule of finite index. Now for all i, $\Phi(\beta_i)H$ is a simple sub-t-module isogenous to H.

As a sub-t-module of a uniformizable abelian t-module, H is also abelian and uniformizable (Remark 2.1.1), and for $W := \mathrm{Lie}(H)$ we have $\mathrm{Exp}_H = \mathrm{Exp}_E|_W$. The action of C on C_∞^d induced by $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B})$ leaves both W and the period lattice $\Lambda_H \subset W$ of H invariant. For each i, $\mathrm{Lie}(\Phi(\beta_i)H) = \boldsymbol{\sigma}_{\mathcal{S}}(\beta_i)W =: W_i$, and $\boldsymbol{\sigma}_{\mathcal{S}}(\beta_i)\Lambda_H$ has finite index in the period lattice of $\Phi(\beta_i)H$.

By a theorem of Anderson [2, Cor. 3.3.6], we know both that Λ spans $\mathrm{Lie}(E)$ over C_∞ and that $\boldsymbol{\sigma}_{\mathcal{S}}(\beta_i)\Lambda_H$ spans W_i over C_∞. As $\boldsymbol{\sigma}_{\mathcal{S}}(\beta_1)\Lambda_H + \dots \boldsymbol{\sigma}_{\mathcal{S}}(\beta_m)\Lambda_H$ has finite index in Λ, we find that $\mathrm{Lie}(E) = W_1 + \cdots + W_m$. Moreover,

$$E = \Phi(\beta_1)H + \cdots + \Phi(\beta_m)H. \tag{2.8}$$

Now as each $\Phi(\beta_i)H$ is simple, we can re-order the β_i so that for some $\ell \leq m$, we have an isogeny

$$E \sim \Phi(\beta_1)H \times \cdots \times \Phi(\beta_\ell)H. \tag{2.9}$$

Thus E is isogenous to H^ℓ, which proves part (a). We will now show that $\ell = m$.

First, the centralizer of $\Phi(\mathsf{K})$ in $\mathrm{End}^0(E)$ is $\Phi(\mathsf{K})$. Indeed, $\boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{K})\Lambda$ is 1-dimensional as a K-vector space. If $f \in \mathrm{End}^0(E)$ commutes with $\Phi(c) \in \Phi(\mathsf{K})$, we see that $(df - \boldsymbol{\sigma}_{\mathcal{S}}(c))|_\Lambda = 0$, which implies that $df = \boldsymbol{\sigma}_{\mathcal{S}}(c)$ and $f = \Phi(c)$.

Now let Z_0 be the center of $\mathrm{End}^0(H)$. From the diagonal mapping $Z_0 \hookrightarrow \mathrm{End}^0(E) \simeq \mathrm{Mat}_{\ell\times\ell}(\mathrm{End}^0(H))$, we see that the image of Z_0 is in the center of $\mathrm{End}^0(E)$ and in particular in the centralizer of $\Phi(\mathsf{K})$. Therefore, $Z_0 \subset \Phi(\mathsf{K})$ by the above paragraph. Thus there is a field $\mathsf{L}_0 \subset \mathsf{K}$ such that $Z_0 = \Phi(\mathsf{L}_0)$.

Let $g = [Z_0 : \Phi(\mathsf{k})]$ and $h^2 = [\mathrm{End}^0(H) : Z_0]$. By the general theory of semi-simple algebras (see [12, pp. 11–12]), since (1) $\mathrm{End}^0(H)$ is a division algebra and (2) $\Phi(\mathsf{K}) \supset \Phi(\mathsf{L}_0)$ is a subfield of $\mathrm{End}^0(E) \simeq \mathrm{Mat}_{\ell\times\ell}(\mathrm{End}^0(H))$ which is its own centralizer, it follows that

$$[K : \mathsf{L}_0] = \ell h.$$

Thus we have

$$\ell \,\mathrm{rank}_{\mathsf{A}}(\Lambda_H) = r = [\mathsf{K} : \mathsf{k}] = [\mathsf{K} : \mathsf{L}_0][\mathsf{L}_0 : \mathsf{k}] = \ell h[\mathsf{L}_0 : \mathsf{k}],$$

and so

$$\mathrm{rank}(H) = \mathrm{rank}_{\mathsf{A}}(\Lambda_H) = gh.$$

Since $gh^2 \mid \mathrm{rank}_{\mathsf{A}}(\Lambda_H)$ by Proposition 2.4.3, it follows that $h = 1$. Therefore $\mathrm{End}^0(H) = \Phi(\mathsf{L}_0)$ and $\mathrm{rank}_{\mathsf{A}}(\Lambda_H) = [\mathsf{L}_0 : \mathsf{k}]$.

On the other hand,

$$\Phi(\mathsf{L}) \subset \mathrm{End}^0(H) = \Phi(\mathsf{L}_0) \subset \Phi(\mathsf{K}).$$

Now $\mathrm{End}(H) \cap \Phi(\mathsf{B})$ stabilizes H and is an order in $\Phi(\mathsf{L}_0)$. Our choice that C be the largest A-subalgebra of B for which $\Phi(\mathsf{C})$ leaves H invariant then implies that $\mathsf{C} \supset \mathrm{End}(H) \cap \Phi(\mathsf{B})$ and so $\mathsf{L} \supset \mathsf{L}_0$. Therefore, $\mathsf{L}_0 = \mathsf{L}$ and

$$\mathrm{rank}_{\mathsf{A}}(\Lambda_H) = \mathrm{rank}_{\mathsf{A}}(\mathsf{C}).$$

From (2.9) we have $\mathrm{rank}_{\mathsf{A}}(\Lambda) = \ell \,\mathrm{rank}_{\mathsf{A}}(\mathsf{C})$, and of course $\mathrm{rank}_{\mathsf{A}}(\Lambda) = \mathrm{rank}_{\mathsf{A}}(\mathsf{B}) = m \,\mathrm{rank}_{\mathsf{A}}(\mathsf{C})$. Hence $\ell = m$ and E is isogenous to H^m.

For part (b) we prove the contrapositive of both directions. If E is not simple, it contains a proper simple sub-t-module H. Picking up the considerations and notation of part (a), we see that $d_0 = d/m$. As the action of C on Λ_H is obtained through $\boldsymbol{\sigma}_{\mathcal{S}}$, we find from part (a) that we can partition

$$\{\sigma_1, \ldots, \sigma_d\} = \bigcup_{i=1}^{m} \{\sigma_{i_1}, \ldots, \sigma_{i_{d_0}}\}$$

such that for all $1 \leq i, j \leq m$ and for all $c \in \mathsf{C}$,

$$(\sigma_{i_1}(c), \ldots, \sigma_{i_{d_0}}(c)) = (\sigma_{j_1}(c), \ldots, \sigma_{j_{d_0}}(c)).$$

Thus for each k, $1 \leq k \leq d_0$, the embeddings $\{\sigma_{i_k}\}_{i=1}^m$ agree on L. Since $m = |\mathrm{Gal}(\mathsf{K}/\mathsf{L})|$, $\{\sigma_{i_k}\}_{i=1}^m$ form a coset of $\mathrm{Gal}(\mathsf{K}/\mathsf{L})$. Thus $\mathcal{S}$ is the union of cosets of $\mathrm{Gal}(\mathsf{K}/\mathsf{L})$, and the criteria of Lemma 2.5.1 are satisfied.

Suppose now that there is a field $\mathsf{L} \subset \mathsf{K}$ satisfying the equivalent criteria in Lemma 2.5.1. By Remark 2.3.1 it suffices to consider the case that the period lattice of E is isomorphic to the maximal order B of K. Let $d_0 := [\mathsf{L}_+ : \mathsf{k}]$, $m := [\mathsf{K} : \mathsf{L}]$, and let C and C_+ be the integral closures of A in L and L_+ respectively. By hypothesis, we know that $\mathcal{S}$ is the union of d_0 cosets of $\mathsf{F} := \mathrm{Gal}(\mathsf{K}/\mathsf{L})$; moreover, suppose without loss of generality that

$$\mathcal{S} = \sigma_1 \mathsf{F} \cup \cdots \cup \sigma_{d_0} \mathsf{F} \tag{2.10}$$

and that we order $\{\sigma_i\}$ so that

$$\sigma_{(j-1)d_0+k} \in \sigma_k \mathsf{F}, \quad 1 \leq j \leq m, \quad 1 \leq k \leq d_0. \tag{2.11}$$

We define Λ_{C} to be the image in $C_\infty^{d_0}$ of C under the conjugate embedding

$$\boldsymbol{\sigma}_{\mathsf{L}} := \sigma_1|_{\mathsf{L}} \oplus \cdots \oplus \sigma_{d_0}|_{\mathsf{L}},$$

as in (2.3) and (2.6), taking $\boldsymbol{\lambda}_{\mathsf{C}} := (1, \ldots, 1)$. By Anderson's Theorem 2.2.1, Λ_{C} is the period lattice of a t-module $H_{\mathsf{C}} = (\mathbb{G}_a^{d_0}, \Phi_{\mathsf{C}})$ of CM-type $(\mathsf{L}, \{\sigma_1|_{\mathsf{L}}, \ldots, \sigma_{d_0}|_{\mathsf{L}}\})$ with complex multiplications by C. Let $\mathrm{Exp}_{\mathsf{C}} : C_\infty^{d_0} \to C_\infty^{d_0}$ be the exponential function of H_{C}.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ be given as in (2.6) so that $\Lambda = \boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B})(\boldsymbol{\lambda})$, and let $\mathsf{M} := \beta_1 \mathsf{C} + \cdots + \beta_m \mathsf{C} \subset \mathsf{B}$ be a direct sum so that M is a free C-module of maximal rank m in inside of B. Let $\Lambda_{\mathsf{M}} := \boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{M})(\boldsymbol{\lambda})$ which has finite index in Λ. Then for $b = \sum_{i=1}^m \beta_i c_i \in \mathsf{M}$, by definition we have

$$\begin{aligned}\boldsymbol{\sigma}_{\mathcal{S}}(b)(\boldsymbol{\lambda}) &= \sum_{i=1}^m \boldsymbol{\sigma}_{\mathcal{S}}(\beta_i c_i)(\boldsymbol{\lambda}) \\ &= \sum_{i=1}^m \begin{pmatrix} \boldsymbol{\sigma}_{\mathsf{L}}(c_i) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\sigma}_{\mathsf{L}}(c_i) \end{pmatrix} \begin{pmatrix} \sigma_1(\beta_i)\lambda_1 \\ \vdots \\ \sigma_d(\beta_i)\lambda_d \end{pmatrix}.\end{aligned} \tag{2.12}$$

Keeping the convention (2.11) in mind, for $1 \leq i, j \leq m$ we let $U_{ji} \in \mathrm{Mat}_{d_0 \times d_0}(C_\infty)$ be the matrix

$$U_{ji} := \begin{pmatrix} \sigma_{(j-1)d_0+1}(\beta_i)\lambda_{(j-1)d_0+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{jd_0}(\beta_i)\lambda_{jd_0} \end{pmatrix},$$

and then set

$$U := (U_{ji}) \in \mathrm{Mat}_{d \times d}(C_\infty).$$

From (2.12) we see that

$$\Lambda_{\mathsf{M}} = \boldsymbol{\sigma}_{\mathcal{S}}(\mathsf{B})(\boldsymbol{\lambda}) = U(\sigma_1(c_1), \ldots, \sigma_{d_0}(c_1), \ldots, \sigma_1(c_m), \ldots, \sigma_{d_0}(c_m))^{tr},$$

where $c_1, \ldots, c_m$ run through C. Since the image of U spans $\mathrm{Lie}(E)$, we know that U is invertible. It follows that for $z_1, \ldots, z_d \in C_\infty$,

$$\mathbf{z} \in \Lambda_{\mathsf{M}} \iff U^{-1}\mathbf{z} \in (\sigma_1(c_1), \ldots, \sigma_{d_0}(c_1), \ldots, \sigma_1(c_m), \ldots, \sigma_{d_0}(c_m))^{tr},$$

where $\mathbf{z} = (z_1, \ldots, z_d)^{tr}$. Let π_i denote projection onto the i-th component of $\boldsymbol{\sigma}_{\mathsf{L}}(\mathsf{C}) \oplus \cdots \oplus \boldsymbol{\sigma}_{\mathsf{L}}(\mathsf{C})$ so that

$$I_d = U \begin{pmatrix} \pi_1 & & 0 \\ & \ddots & \\ 0 & & \pi_{d_0} \end{pmatrix} U^{-1}. \tag{2.13}$$

Now define an analytic map $e_{\mathsf{M}} : C_\infty^d \to C_\infty^d$ by

$$e_{\mathsf{M}}(\mathbf{z}) := U \begin{pmatrix} \mathrm{Exp}_{\mathsf{C}}(\pi_1 U^{-1}(\mathbf{z})) \\ \vdots \\ \mathrm{Exp}_{\mathsf{C}}(\pi_{d_0} U^{-1}(\mathbf{z})) \end{pmatrix}. \tag{2.14}$$

The function $e_{\mathsf{M}}(\mathbf{z})$ then (1) is entire; (2) vanishes exactly on Λ with simple zeros; (3) as a power series in $z_1, \ldots, z_d$ satisfies $\partial e_{\mathsf{M}}(\mathbf{z}) = I_d$ according to (2.13); and (4) inherits a functional equation from $\mathrm{Exp}_{\mathsf{C}}$. These properties make $e_{\mathsf{M}}(\mathbf{z})$ the exponential function of the t-module $E_{\mathsf{M}} = (\mathbb{G}_a^d, \Phi_{\mathsf{M}})$ where $\Phi_{\mathsf{M}} : \mathsf{A} \to \mathrm{Mat}_{d\times d}(C_\infty\{\tau\})$ is given by

$$\Phi(a) = U \begin{pmatrix} \Phi_{\mathsf{C}}(a) & & 0 \\ & \ddots & \\ 0 & & \Phi_{\mathsf{C}}(a) \end{pmatrix} U^{-1}.$$

As $(\pi_i U^{-1}(U\mathbf{z}))_i = (\mathbf{z}_1, \ldots, \mathbf{z}_m)$ for arbitrary $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_m) \in (C_\infty^{d_0})^m$, by (2.14) this t-module is uniformizable, since H_{C} is. Moreoever, it is abelian since H_{C} is. By construction E_{M} is isomorphic to H_{C}^m.

Now $\Lambda_{\mathsf{M}} \subset \Lambda$ has finite index. By Lemma 2.1.2 there is a uniformizable abelian t-module E_Λ isogenous to E_{M}, which has Λ as its period lattice. However, uniformizable abelian t-modules are determined by their period lattices by a theorem of Anderson [2, Cor. 2.12.2], and so $E_\Lambda = E$. Therefore, E is isogenous to H_{C}^m and so contains a sub-t-module isogenous to H_{C}. □

Remark. The identity (2.14) explicitly determines the way E_{M} decomposes as a product H_{C}^m. In practice, if $E = E_{\mathsf{M}}$, this is a fruitful method for determining the exponential functions of such non-simple t-modules of CM-type, as shown by the examples in Section 6.

Theorem 2.5.3. *Let E_1 and E_2 be t-modules of CM-types $(\mathsf{K}_1, \mathcal{S}_1)$ and $(\mathsf{K}_2, \mathcal{S}_2)$ respectively. There exists a non-zero t-module morphism $E_1 \to E_2$ if and only if there is a subfield $\mathsf{L} \subset \mathsf{K}_1 \cap \mathsf{K}_2$ and an automorphism $\rho \in \mathrm{Gal}(\mathsf{L}/\mathsf{k})$ such that L simultaneously satisfies the criteria of Lemma 2.5.1 for both K_1 and K_2 and $\mathcal{S}_1|_{\mathsf{L}} = (\mathcal{S}_2|_{\mathsf{L}})\rho$.*

Proof. Suppose E_1 is isogenous to $H_1^{m_1}$ and E_2 is isogenous to $H_2^{m_2}$ with H_1 and H_2 both simple. From Theorem 2.5.2a it follows that the group $\mathrm{Hom}(E_1, E_2)$ of t-module morphisms is non-trivial if and only if H_1 and H_2 are isogenous. Because for the concerns of this theorem we fix our sub-t-modules only up to isogeny, by Remark 2.3.1 we can assume that H_1 and H_2 have period lattices isomorphic to the full rings of integers in their respective CM-fields.

Now suppose a subfield $\mathsf{L} \subset \mathsf{K}_1 \cap \mathsf{K}_2$ exists as in the statement of the proposition. Let $\mathcal{S}_1|_{\mathsf{L}} = \{\sigma_1, \ldots, \sigma_d\}$ and $\mathcal{S}_2|_{\mathsf{L}} = \{\tau_1, \ldots, \tau_d\}$. Because $\mathcal{S}_1|_{\mathsf{L}} = (\mathcal{S}_2|_{\mathsf{L}})\rho$, we can reorder the coordinates of either H_1 or H_2 so that

$$\sigma_i|_{\mathsf{L}} = (\tau_i|_{\mathsf{L}})\rho, \quad 1 \le i \le d. \tag{2.15}$$

Let C be the ring of integers in L. We can assume without loss of generality that for some $\boldsymbol{\lambda} \in C_\infty^d$ the period lattices Λ_1 and Λ_2 of H_1 and H_2 are

$$\Lambda_1 = \boldsymbol{\sigma}_{\mathcal{S}_1|_{\mathsf{L}}}(\mathsf{C})(\boldsymbol{\lambda}) \quad \text{and} \quad \Lambda_2 = \boldsymbol{\sigma}_{\mathcal{S}_2|_{\mathsf{L}}}(\mathsf{C})(\boldsymbol{\lambda}).$$

By (2.15) it then follows that $\Lambda_1 = \Lambda_2$, and so, as uniformizable abelian t-modules are determined by their period lattices, H_1 and H_2 are the same (by [2, Cor. 2.12.2]).

For the other direction suppose that there is an isogeny $\psi : H_1 \to H_2$. As t-modules we have an isomorphism

$$\widetilde{\psi} : H_1/(\ker\psi) \xrightarrow{\sim} H_2,$$

as constructed in Section 2.1.3, and further an isomorphism

$$\widetilde{\psi}_* : \mathrm{End}(H_1/(\ker\psi)) \xrightarrow{\sim} \mathrm{End}(H_2)$$

given by $\widetilde{\psi}_*(f) = \widetilde{\psi} \circ f \circ \widetilde{\psi}^{-1}$. Thus H_1 and H_2 have the same CM-field L. By Corollary 2.4.4 and our assumptions in the first paragraph we have

$$\mathrm{End}(H_1/(\ker\psi)) \simeq \mathrm{End}(H_2) \simeq \mathsf{C},$$

where C is the ring of integers in L. Let $\Phi_1 : \mathsf{C} \to \mathrm{End}(H_1/(\ker\psi))$ and $\Phi_2 : \mathsf{C} \to \mathrm{End}(H_2)$ be choices of such isomorphisms which induce the conjugate actions on their tangent spaces as in (2.5). Therefore for each $c \in \mathsf{C}$ we have

(2.16)
$$\begin{array}{ccc}
\mathrm{Lie}(H_1/(\ker\psi)) & \xrightarrow{d\widetilde{\psi}} & \mathrm{Lie}(H_2) \\
{\scriptstyle d\Phi_1(c) = \boldsymbol{\sigma}_{\mathcal{S}_1|_{\mathsf{L}}}(c)} \downarrow & & \downarrow {\scriptstyle d(\widetilde{\psi}_*(\Phi_1(c)))} \\
\mathrm{Lie}(H_1/(\ker\psi)) & \xrightarrow[d\widetilde{\psi}]{} & \mathrm{Lie}(H_2).
\end{array}$$

Because $\widetilde{\psi}_* \circ \Phi_1 : \mathsf{C} \to \mathrm{End}(H_2)$ is an isomorphism, $\Phi_2^{-1} \circ \widetilde{\psi}_* \circ \Phi_1$ is an automorphism of C which fixes A. So it must be the case for some $\rho \in \mathrm{Gal}(\mathsf{L}/\mathsf{k})$ that

$$\widetilde{\psi}_* \circ \Phi_1 = \Phi_2 \circ \rho.$$

We set $d(\widetilde{\psi} \circ \Phi_1) = d(\Phi_2 \circ \rho) = \boldsymbol{\sigma}_{\mathcal{S}_2|_{\mathsf{L}}} \circ \rho$ in (2.16), and since similar matrices have the same eigenvalues, we conclude that $\mathcal{S}_1|_{\mathsf{L}} = (\mathcal{S}_2|_{\mathsf{L}})\rho$. $\square$

3. Biderivations and Quasi-Periodic Extensions of t-Modules

The theory of biderivations and quasi-periodic extensions for Drinfeld modules was developed by P. Deligne, Anderson, and Yu. E.-U. Gekeler gave an alternate approach to the de Rham isomorphism in the very accessible source [10]. Anderson suggested the relevance of quasi-periods for obtaining Gamma values at those rational points which are not the ratio of monic polynomials to Thakur, who passed the hint along to us some years later. Here we extend much of [4] from the setting of A-Drinfeld modules to that of arbitrary t-modules.

For the remainder of Section 3 we fix a t-module $E = (\Phi, \mathbb{G}_a^d)$.

3.1. **Biderivations.** A Φ-*biderivation* is an $\mathbb{F}_q$-linear map $\boldsymbol{\delta} : \mathsf{A} \to \tau M(E)$ satisfying the product formula that, for all a, $b \in \mathsf{A}$,

$$\boldsymbol{\delta}(ab) = \iota(a) \cdot \boldsymbol{\delta}(b) + \boldsymbol{\delta}(a)\Phi(b).$$

Lemma 3.1.1. *Let $m \in \tau M(E)$, i.e. m is any element of the t-motive $M(E)$ associated to E such that m has no τ^0 terms:*

$$m \in \mathrm{Hom}_L^q(E, \mathbb{G}_a), \quad dm = 0.$$

Then the assignment $t \mapsto \boldsymbol{\delta}(t) := m$ induces a Φ-biderivation $\boldsymbol{\delta}_m$.

Proof. By $\mathbb{F}_q$-linearity, we need only check that the "competing" expressions given by applying the product formula inductively to different factorizations

$$t^{m_1}t^{n_1} = t^{m_2}t^{n_2},\ m_i, n_i \in \mathbb{Z}_{>0},$$

are equal. This verification is straightforward. □

This lemma gives a natural isomorphism between the L-vector spaces $\mathrm{Der}(\Phi)$ and $\tau M(E)$. Certain Φ-biderivations can be given algebraically in terms of Φ using the identification $M(E) \simeq (L\{\tau\})^d$. Set

$$N^\perp := N^\perp(L) := \{V \in \mathrm{Mat}_{1\times d}(L) : VN = 0\},$$

where N is the nilpotent part of $d\Phi(t) = \theta I_d + N$.

Let $\boldsymbol{U} = (U_1, \ldots, U_d) \in (L\{\tau\})^d$ with $d\boldsymbol{U} \in N^\perp(L)$, where $d\boldsymbol{U} = (dU_1, \ldots, dU_d)$ denotes the vector of coefficients of τ^0 in $\boldsymbol{U}$. We define $\boldsymbol{\delta}^{(\boldsymbol{U})} : \mathsf{A} \to M(E)$ via

$$\boldsymbol{\delta}^{(\boldsymbol{U})}(a) := \boldsymbol{U}\Phi(a) - \iota(a)\boldsymbol{U}, \tag{3.1}$$

for every $a \in \mathsf{A}$. The condition $d\boldsymbol{U} \in N^\perp$ is equivalent to saying that $d\boldsymbol{\delta}^{(\boldsymbol{U})}(t) = 0$, i.e. that $\boldsymbol{\delta}^{(\boldsymbol{U})}(t) \in \tau M(E)$.

Since $\boldsymbol{\delta}^{(\boldsymbol{U})}(ab) = \boldsymbol{U}\Phi(ab) - \iota(ab)\boldsymbol{U} = \iota(a)(\boldsymbol{U}\Phi(b) - \iota(b)\boldsymbol{U}) + (\boldsymbol{U}\Phi(a) - \iota(a)\boldsymbol{U})\Phi(b)$, $\boldsymbol{\delta}^{(\boldsymbol{U})}$ is indeed a Φ-biderivation. Such Φ-biderivations will be called *inner* and they constitute an L-vector space which we denote $\mathrm{Der}_{in}(\Phi)$. Note further that, in terms of the t-motive $M(E)$, we are setting $\boldsymbol{\delta}^{(\boldsymbol{U})}(t) = (t-\theta)\boldsymbol{U}$.

Lemma 3.1.2. *If $M = M(E)$ is a torsion-free t-motive over L, then, as L-vector spaces,*

$$M_{in} := (t-\theta)(\tau M + N^\perp \tau^0) \simeq \mathrm{Der}_{in}(\Phi)$$

via the natural isomorphism

$$(t-\theta)\boldsymbol{U} \mapsto \boldsymbol{\delta}^{(\boldsymbol{U})}.$$

Proof. It is clear that for $\boldsymbol{U} \in M$,

$$(t-\theta)\boldsymbol{U} \in \mathrm{Der}_{in}(\Phi) \Longleftrightarrow \boldsymbol{U} \in \tau M + N^\perp \tau^0.$$

Since M is torsion-free, multiplication by $t - \theta$ is injective. □

Definition. Several other distinguished subspaces of $\mathrm{Der}(\Phi)$ and quotient spaces will play a role in our discussion:

$$\mathrm{Der}_0(\Phi) := \{\boldsymbol{\delta}^{(\boldsymbol{U})} : \boldsymbol{U} \in N^\perp(L)\tau^0\} = \{\boldsymbol{\delta}^{(\boldsymbol{U})} \in \mathrm{Der}_{in}(\Phi) : \boldsymbol{U} \in L^d\tau^0\}$$

$$\mathrm{Der}_{si}(\Phi) := \{\boldsymbol{\delta}^{(\boldsymbol{U})} : \boldsymbol{U} \in \tau M\} \qquad \text{(strictly inner)}$$

$$H_{DR}(\Phi) := \mathrm{Der}(\Phi)/\,\mathrm{Der}_{si}(\Phi) \qquad \text{(de Rham)}$$

$$H_{sr}(\Phi) := \mathrm{Der}(\Phi)/\,\mathrm{Der}_{in}(\Phi) \qquad \text{(strictly reduced)}$$

The phrases after the definitions indicate the names for the type of biderivation involved and thus account for the subscripts employed.

Proposition 3.1.3. *For any t-motive $M = M(E)$ over L, where $E = (\Phi, \mathbb{G}_a^d)$,*

$$\mathrm{Der}_{in}(\Phi) = \mathrm{Der}_0(\Phi) \oplus \mathrm{Der}_{si}(\Phi)$$
$$H_{DR}(\Phi) \simeq \mathrm{Der}_0(\Phi) \oplus H_{sr}(\Phi)$$

and, if M is abelian,

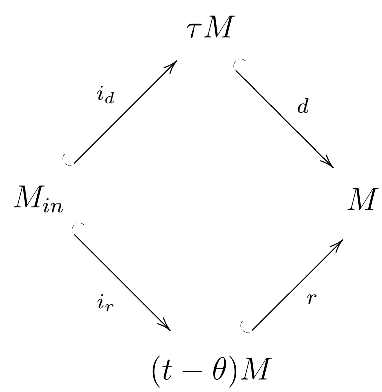
$$\dim_L \mathrm{Der}_0(\Phi) = d - \operatorname{rank} N \tag{3.2}$$

$$\dim_L H_{DR}(\Phi) = r, \tag{3.3}$$

$$\dim_L H_{sr}(\Phi) = r - d + \operatorname{rank} N \tag{3.4}$$

where $d = \dim E = \operatorname{rank}_{L\{\tau\}} M$, $r = \operatorname{rank} E = \operatorname{rank}_{L[t]} M$, and $d\Phi(t) = \theta I_d + N$.

Proof. The direct sum decompositions are immediate. By definition, $\dim_L \mathrm{Der}_0(\Phi) = \dim_L N^\perp = d - \operatorname{rank} N$. To calculate the dimension of $H_{sr}(\Phi)$, we appeal to the following diagram, where here we assume that E is abelian of rank r:



The labels indicate the L-dimensions of the quotients. By Lemma 3.1.2, i_r is the codimension of $\tau M + N^\perp\tau^0$ in M. Thus $i_r = \operatorname{rank} N$ and $\dim_L H_{sr}(\Phi) = i_d = r - d + \operatorname{rank} N$. □

3.2. **Quasi-Periodic Functions.** In this section we investigate quasi-periodic functions and quasi-periods associated to Φ-biderivations. The reader is directed to Gekeler [10] for a historical motivation for this and related terminology.

Proposition 3.2.1. *Given a Φ-biderivation $\boldsymbol{\delta}$ defined over L, there is a unique entire $\mathbb{F}_q$-linear function $F_{\boldsymbol{\delta}} : C_\infty^d \to C_\infty$ satisfying:*

$$F_{\boldsymbol{\delta}}(d\Phi(a)\mathbf{z}) = \iota(a)F_{\boldsymbol{\delta}}(\mathbf{z}) + \boldsymbol{\delta}(a)\operatorname{Exp}(\mathbf{z}), \tag{3.5}$$

$$F_{\boldsymbol{\delta}}(\mathbf{z}) \equiv 0 \pmod{\mathbf{z}^q}, \tag{3.6}$$

where the latter condition means that every non-zero term in the power series $F_{\boldsymbol{\delta}}(\mathbf{z})$ is that of a monomial $z_i^{q^h}$ with $h > 0$. Furthermore, $F_{\boldsymbol{\delta}}(\mathbf{z})$ has coefficients from L.

Proof. Write $\operatorname{Exp}(\mathbf{z}) = (e_1(\mathbf{z}), \dots, e_d(\mathbf{z}))$, $\boldsymbol{\delta}(t) = (\delta_1(t), \dots, \delta_d(t))$, with the variables chosen so that $N = (n_{ij})$ is upper triangular. If

$$F_{\boldsymbol{\delta}}(\mathbf{z}) = \sum_h \mathbf{c}_h \cdot \mathbf{z}^{q^h} = \sum_h c_{h1} z_1^{q^h} + \cdots + c_{hd} z_d^{q^h},$$

then we can equate coefficients to find the equality

$$(\theta^{q^h} - \theta)c_{h1}z_1^{q^h} = \text{Term involving } z_1^{q^h} \text{ in } \sum \delta_i(t)e_i(\mathbf{z}).$$

It shows that the c_{h1} is uniquely determined and that the terms $c_{h1}z_1^{q^h} \to 0$ as $h \to \infty$, for any fixed value of z_1. Therefore, for any fixed values of z_2, the terms $c_{h1}(z_2n_{12})^{q^h}$ tend to zero as $h \to \infty$ in the following equality:

$$(\theta^h - \theta)c_{h2}z^{q^h} + c_{h1}n_{12}^{q^h}z_2^{q^h} = \text{Term involving } z_2^{q^h} \text{ in } \sum \boldsymbol{\delta}_i(t)e_i(\mathbf{z}).$$

Consequently, we find that the terms $c_{h2}z_2^{q^h}$ tend toward zero as $h \to \infty$ for any fixed value of z_2. Iterating this argument, we find that each series

$$\sum_h c_{ih}z_i^{q^h}$$

is uniquely determined and entire. Thus $F_{\boldsymbol{\delta}}(\mathbf{z})$ is uniquely determined and everywhere convergent.

The $\mathbb{F}_q$-linearity of $F_{\boldsymbol{\delta}}(\mathbf{z})$ and the following recursive calculation shows that $F_{\boldsymbol{\delta}}(\mathbf{z})$ satisfies the required functional equation with respect to $\boldsymbol{\delta}$:

$$\begin{aligned} F_{\boldsymbol{\delta}}(d\Phi(t^{i+1})\mathbf{z}) &= \theta F_{\boldsymbol{\delta}}(d\Phi(t^i)\mathbf{z}) + \boldsymbol{\delta}(t)\operatorname{Exp}(d\Phi(t^i)\mathbf{z}) \\ &= \theta(t^i F_{\boldsymbol{\delta}}(\mathbf{z}) + \boldsymbol{\delta}(t^i)\operatorname{Exp}(\mathbf{z})) + \boldsymbol{\delta}(t)\Phi(t^i)\operatorname{Exp}(\mathbf{z}) \\ &= \theta^{i+1}F_{\boldsymbol{\delta}}(\mathbf{z}) + \boldsymbol{\delta}(t^{i+1})\operatorname{Exp}(\mathbf{z}). \end{aligned}$$

□

The unique $\mathbb{F}_q$-linear function given by Proposition 3.2.1 is said to be the *quasi-periodic function associated to* $\boldsymbol{\delta}$.

It is easy to check from the unicity of $F_{\boldsymbol{\delta}}$ that, if $\boldsymbol{\delta} = \ell_1\boldsymbol{\delta}_1 + \ell_2\boldsymbol{\delta}_2$, with $\ell_1, \ell_2 \in L$, then

$$F_{\boldsymbol{\delta}}(\mathbf{z}) = \ell_1 F_{\boldsymbol{\delta}_1}(\mathbf{z}) + \ell_2 F_{\boldsymbol{\delta}_2}(\mathbf{z}).$$

Remark. Note that, since the image of $\operatorname{Exp}(\mathbf{z})$ is dense in $\mathbb{G}_a^d(C_\infty)$, if the biderivation $\boldsymbol{\delta}$ is non-zero, then all functions satisfying the functional equation (3.5) of Proposition 3.2.1 are non-zero, even if we allow solutions which violate (3.6). The solutions to that relaxed functional equation comprise the set $F_{\boldsymbol{\delta}}(\mathbf{z}) + N^{\perp}(L) \cdot \mathbf{z}$, where $F_{\boldsymbol{\delta}}(\mathbf{z})$ is the quasi-periodic function of $\boldsymbol{\delta}$.

Proposition 3.2.2. *The quasi-periodic function related to* $\boldsymbol{\delta}^{(\boldsymbol{U})} \in \operatorname{Der}_{in}(\Phi)$ *is*

$$F^{(\boldsymbol{U})}(\mathbf{z}) := \boldsymbol{U}\operatorname{Exp}(\mathbf{z}) - d\boldsymbol{U}\mathbf{z}.$$

Proof. Note that $F^{(\boldsymbol{U})}(\mathbf{z})$ has no linear terms and that the functional equation holds:

$$\begin{aligned} F^{(\boldsymbol{U})}(d\Phi(t)\mathbf{z}) &= \boldsymbol{U}\operatorname{Exp}(d\Phi(t)\mathbf{z}) - d\boldsymbol{U}d\Phi(t)\mathbf{z} \\ &= \theta(\boldsymbol{U}\operatorname{Exp}(\mathbf{z}) - d\boldsymbol{U}\mathbf{z}) + (\boldsymbol{U}\Phi(t) - \theta\boldsymbol{U})\operatorname{Exp}(\mathbf{z}). \end{aligned}$$

□

If $\boldsymbol{\lambda} \in \Lambda := \operatorname{Ker}\operatorname{Exp}$ is a period of of E and $\boldsymbol{\delta} \in \operatorname{Der}(\Phi)$, then $\eta := \eta_\delta(\boldsymbol{\lambda}) := F_{\boldsymbol{\delta}}(\boldsymbol{\lambda})$ is called the *quasi-period of* $\boldsymbol{\delta}$ *corresponding to* $\boldsymbol{\lambda}$. The following result is an immediate corollary of Proposition 3.2.2.

Corollary 3.2.3. *The quasi-period* $\eta_{\boldsymbol{\delta}^{(U)}}(\boldsymbol{\lambda})$ *associated to the inner biderivation* $\boldsymbol{\delta} := \boldsymbol{\delta}^{(\boldsymbol{U})}$ *is*

$$\eta_{\boldsymbol{\delta}^{(U)}}(\boldsymbol{\lambda}) = d\boldsymbol{U} \cdot \boldsymbol{\lambda}.$$

Therefore the quasi-periods of inner biderivations defined over L are L-linear combinations of the coordinates of the corresponding periods.

3.3. **Quasi-Periodic Extensions.** Let $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_j$ be Φ-biderivations. Then define the matrix

$$\Psi(t) := \left(\begin{array}{c|c} \Phi(t) & 0 \\ \hline \boldsymbol{\delta}_1(t) & \\ \vdots & \theta I_j \\ \boldsymbol{\delta}_j(t) & \end{array}\right),$$

where $\boldsymbol{\delta}_i(t)$ is situated directly beneath $\Phi(t)$ in the $(d+i)$th row of $\Psi(t)$, and the unique following non-zero entry in that row is in the $(d+i)$th column, i.e. in the ith column following the entries for $\boldsymbol{\delta}_i(t)$. Here and below we omit the τ^0 from the linear terms for easier reading of matrices. In addition, define the entire mapping

$$\mathrm{Exp}_\Psi : C_\infty^{d+j} \to C_\infty^{d+j}$$

via

$$\mathrm{Exp}_\Psi(\mathbf{z}, \mathbf{u}) := \left(\mathrm{Exp}(\mathbf{z}), u_1 + F_1(\mathbf{z}), \ldots, u_j + F_j(\mathbf{z})\right)^{tr},$$

where for simplicity we have written F_i for $F_{\boldsymbol{\delta}_i}$.

Proposition 3.3.1. *In the situation of the preceding paragraph,* $Q := (\Psi, \mathbb{G}_a^{d+j})$ *is a* t*-module with exponential function* $\mathrm{Exp}_Q = \mathrm{Exp}_\Psi$ *and with periods*

$$(\boldsymbol{\lambda}, -\eta_1(\boldsymbol{\lambda}), \ldots, -\eta_j(\boldsymbol{\lambda})),$$

where $\eta_i(\boldsymbol{\lambda})$ *is the quasi-period of* $\boldsymbol{\delta}_i$ *corresponding to the period* $\boldsymbol{\lambda} \in C_\infty^d$.

When $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_j$ represent L-linearly independent classes in $H_{sr}(\Phi)$, we call the corresponding extension Q of E *strictly quasi-periodic extension.*

Proof. That Ψ defines a t-module follows from the hypothesis that Φ does and that

$$d\Psi(t) = \begin{pmatrix} d\Phi(t) & 0 \\ 0 & \theta I_j \end{pmatrix},$$

where I_j is the $j \times j$ identity matrix.

That Exp_Ψ satisfies the appropriate functional equation

$$\mathrm{Exp}_\Psi(d\Psi(t)(\mathbf{z}, \mathbf{u})) = \Psi(t)\, \mathrm{Exp}_\Psi(\mathbf{z}, \mathbf{u})$$

follows from the functional equations for Exp_Φ and for the F_i. Moreover, since the $F_i(\mathbf{z})$ have zero linear terms, the linear terms of $\mathrm{Exp}_\Psi(\mathbf{z}, \mathbf{u})$ are precisely $(\mathbf{z}, \mathbf{u})^{tr}$, as required for the exponential function of a t-module. □

Thus we find that Ψ gives an extension of the t-module E by the basic t-module $\mathbb{G}_a^j$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_a^j(C_\infty) & \longrightarrow & Q & \longrightarrow & E & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{Exp}_\Psi} & & \uparrow{\scriptstyle \mathrm{Exp}_\Phi} & & \\ 0 & \longrightarrow & \mathbb{G}_a^j(C_\infty) & \longrightarrow & \mathrm{Lie}(Q) & \longrightarrow & \mathrm{Lie}(E) & \longrightarrow & 0. \end{array}$$

This is a generalization of the one-dimensional (Drinfeld) case and an exact analogue of the extensions of an elliptic curve $\mathcal{E}$ by the additive group $\mathbb{G}_a$. In the latter case we have the exponential maps

$$\begin{array}{lcr} \mathrm{Exp}_\Phi & \longleftrightarrow & z \mapsto (\wp(z), \wp'(z), 1) \\ \mathrm{Exp}_\Psi & \longleftrightarrow & (u, z) \mapsto (u + b\zeta(z), \wp(z), \wp'(z), 1), \end{array}$$

involving the Weierstrass quasi-elliptic function ζ and a constant $b \in \mathbb{C}$ (see [22]). The various choices of the constant b classify the possible extensions, with $b = 0$ giving the trivial, i.e. split, extension. Thus $\mathrm{Ext}(\mathcal{E}, \mathbb{G}_a) \simeq \mathbb{C}$.

Remark. The quasi-period extension Q is uniformizable if and only if E is uniformizable. Moreover notice that Q is never abelian for $j > 0$, since the $\Psi(t)$ action on the latter j coordinates is multiplication by the scalar θ.

The *leitmotiv* of the remainder of this section is that quasi-periodic extensions depend essentially only on the strictly reduced cohomology classes involved. The first step is given by the following:

Proposition 3.3.2. *Let $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_j$ and $\boldsymbol{\delta}'_1, \ldots, \boldsymbol{\delta}'_j$ be Φ-biderivations. If both sets generate the same subspace of $H_{sr}(\Phi)$, then the corresponding quasi-periodic extensions are isomorphic.*

Proof. This proposition follows from the following remark, whose verification is immediate. The first identity there shows that the isomorphism class of the extension is independent of the representatives chosen for the classes in $H_{sr}(\Phi)$. The second shows that the isomorphism class is independent of the generators chosen for span in $H_{sr}(\Phi)$. □

Lemma 3.3.3. *If $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2$ are Φ-biderivations and $\boldsymbol{U} \in M(E)$ with $d\boldsymbol{U} \in N^\perp$, then*

$$\begin{pmatrix} I_d & 0_d \\ \boldsymbol{U} & 1 \end{pmatrix} \begin{pmatrix} \Phi(t) & 0_d \\ \boldsymbol{\delta}_1(t) & \theta \end{pmatrix} \begin{pmatrix} I_d & 0_d \\ -\boldsymbol{U} & 1 \end{pmatrix} = \begin{pmatrix} \Phi(t) & 0_d \\ \boldsymbol{\delta}_1(t) + \boldsymbol{\delta}^{(\boldsymbol{U})}(t) & \theta \end{pmatrix}$$

$$\begin{pmatrix} I_d & 0_d & 0_d \\ 0_d^{tr} & 1 & 0 \\ 0_d^{tr} & c & 1 \end{pmatrix} \begin{pmatrix} \Phi(t) & 0_d & 0_d \\ \boldsymbol{\delta}_1(t) & \theta & 0 \\ \boldsymbol{\delta}_2(t) & 0 & \theta \end{pmatrix} \begin{pmatrix} I_d & 0_d & 0_d \\ 0_d^{tr} & 1 & 0 \\ 0_d^{tr} & -c & 1 \end{pmatrix} = \begin{pmatrix} \Phi(t) & 0_d & 0_d \\ \boldsymbol{\delta}_1(t) & \theta & 0 \\ \boldsymbol{\delta}_2(t) + c\boldsymbol{\delta}_1(t) & 0 & \theta \end{pmatrix},$$

where 0_d denotes the zero vector of length d.

One important consequence of this lemmas is the following remark:

Corollary 3.3.4. *Let Q be the extension of E associated to the Φ-biderivation $\boldsymbol{\delta}$. Then the extension*

$$0 \to \mathbb{G}_a \to Q \to E \to 0$$

splits if and only if $\boldsymbol{\delta}$ is inner.

Proof. If $\boldsymbol{\delta} = \boldsymbol{\delta}^{(\boldsymbol{U})}$ is inner, then choose $\boldsymbol{\delta}_1 = 0$ in the first part of the preceding lemma.

Similarly, if there is an self-isogeny Θ of Q such that

$$\Theta\Psi(t) = \begin{pmatrix} \Phi(t) & 0_d \\ 0_d^{tr} & \theta \end{pmatrix} \Theta,$$

then comparing entries first in the lower right-hand corner gives that the lower right-hand entry c in Θ satisfies $c\theta = \theta c$, i.e. $c \in L$. Then considering entries along the bottom row shows that, up to a non-zero scalar multiple c,

$$c\boldsymbol{\delta}(t) + \boldsymbol{U}\Phi(t) = \theta\boldsymbol{U}.$$

where $(\boldsymbol{U}; c\tau^0)$ is the bottom row of Θ. Now if $c = 0$, then $(t-\theta)\boldsymbol{U} = 0$. However, since the t-motive $M(Q)$ is a free t-module, we would have $\boldsymbol{U} = 0$. Thus the bottom line of Θ would consist of zeros. In that case, Θ would have an infinite kernel and could not be an isogeny. Therefore $c \neq 0$, and $\boldsymbol{\delta}$ is inner, as claimed. ☐

3.4. **Minimality of Quasi-Periodic Extensions.** Let Δ be a surjective morphism from the uniformizable t-module $(\Upsilon, \mathbb{G}_a^e)$ to the uniformizable t-module $(\Phi, \mathbb{G}_a^d)$. We say that $(\Upsilon, \mathbb{G}_a^e)$ is a *minimal extension* of $(\Phi, \mathbb{G}_a^d)$ if no proper sub-t-module of $(\Upsilon, \mathbb{G}_a^e)$ surjects onto $(\Phi, \mathbb{G}_a^d)$. Intuitively this places $(\Upsilon, \mathbb{G}_a^e)$ at the opposite extreme from a split extension. That intuition will be made precise in this subsection.

Proposition 3.4.1. *Let Q be the extension associated to Φ-biderivations $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$. Then the following are equivalent:*

(a) *The biderivations $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$ represent linearly independent classes in $H_{sr}(\Phi)$.*

(b) *$Q = (\Psi, \mathbb{G}_a^{d+j})$ is a minimal extension of E.*

Proof. (a) $\Rightarrow$ (b). We adopt the notation of Section 3.3 for our strictly reduced quasi-periodic extension. The plan is to start with a non-trivial algebraic relation on a proper sub-t-module H of Q and conclude that the image of H lies in a proper sub-t-module of E. Choose coordinates for E so that $d\Phi(t)$ is upper-triangular.

Assume that we have a non-trivial relation holding on the coordinates of H, which is smallest with respect to the reverse lexicographical ordering on monomials in $x_1, \dots, x_d, u_1, \dots, u_j$. Since the underlying group is $\mathbb{F}_q$-linear, this minimal relation has the following form for all $(\mathbf{x}, \mathbf{u}) \in H$ (see [13]):

$$R(\mathbf{x}, \mathbf{u}) = R_1(x_1) + \cdots + R_d(x_d) + S_1(u_1) + \cdots + S_j(u_j) = 0, \tag{3.7}$$

in which all the R_i, $S_i \in L\{\tau\}$. Now we make a series of observations based on the fact that H is a sub-t-module, i.e. that, for all $(\mathbf{x}, \mathbf{u}) \in H$,

$$R \circ \Psi(t)(\mathbf{x}, \mathbf{u}) = R_1'(x_1) + \cdots + R_d'(x_d) + S_1'(u_1) + \cdots + S_j'(u_j) = 0 \tag{3.8}$$

as well. We show by contradiction that the variables of $\mathbf{u}$ are not involved in this minimal relation.

Since the relation (3.7) is minimal and the effect of $\Psi(t)$ on the variables $\mathbf{u}$ is multiplication by θ (plus a sum affecting the variables of $\mathbf{x}$), we apply $\Psi(t)$ to obtain another algebraic relation and, on comparing maximal monomials with respect to our ordering, we conclude that, if the variables $\mathbf{u}$ were involved in (3.7), then for each i

$$S_i(\theta u_i) = \theta^{q^h} S_i(u_i),$$

for a fixed h. Thus, since $\theta \neq \overline{\mathbb{F}}_q$, each S_i is a monomial of degree q^h. However L is a perfect field, and we can write

$$S_1(u_1) + \cdots + S_j(u_j) = (s_1u_1 + \cdots + s_ju_j)^{q^h}, \tag{3.9}$$

with the $s_i \in L$.

Again comparing terms in the relations (3.7), (3.8) and (3.9), but this time for the variables in $\mathbf{x}$, we find that

$$\theta^{q^h}\mathbf{R}\mathbf{x} = \mathbf{R}\Phi(t)\mathbf{x} + \tau^h\boldsymbol{\delta}(t)\mathbf{x}, \tag{3.10}$$

where $\mathbf{R} = (R_1, \ldots, R_d)$ and $\boldsymbol{\delta} := s_1\boldsymbol{\delta}_1 + \cdots + s_j\boldsymbol{\delta}_j$. In particular, for each variable x_ℓ,

$$\theta^{q^h}R_\ell x_\ell = \sum R_i\phi_{i,\ell}(t)x_\ell + s_i^{q^h}\tau^h\delta_{i,\ell}(t)x_\ell,$$

where $\Phi(t) = (\phi_{i,\ell}(t))$ and $\boldsymbol{\delta}_i(t) = (\delta_{i,1}(t), \ldots, \delta_{i,j}(t)) \in \tau M$.

Assume for the moment that $h \neq 0$. Let $R_\ell = r_\ell\tau^0 +$ higher degree terms, $r_\ell \in L$. Notice now that the $\delta_{i,\ell}(t)$ lack linear terms. Since $d\Phi(t) = \theta I_d + N$ is upper-triangular, $\phi_{i,1}(t)$ has no linear terms unless $i = 1$. From (3.10) we see that

$$\theta^{q^h}r_1 = r_1\theta.$$

Because $h \neq 0$, it follows that $r_1 = 0$, i.e. R_1 does not have a linear term. Similarly from (3.10) we see that

$$\theta^{q^h}r_2 = r_1\, d\phi_{1,2}(t) + r_2\theta.$$

Since $r_1 = 0$, we see as above that $r_2 = 0$. Proceeding by induction we find that $r_\ell = 0$ for $\ell = 1, \ldots, d$, and so none of the R_ℓ involve linear terms. But in that case, since the R_ℓ are $\mathbb{F}_q$-linear, the relation (3.7) would not be minimal; we could extract the qth root and have a equation of smaller degree for elements of H.

Therefore we are reduced to the case $h = 0$, and we find that (3.10) has the form

$$\theta\mathbf{R} = \mathbf{R}\Phi(t) + \boldsymbol{\delta}(t).$$

However this means that $-\boldsymbol{\delta}$ and thus $\boldsymbol{\delta}$ are inner, whereas the given $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_j$ are L-linearly independent modulo the inner biderivations. Consequently $\boldsymbol{\delta} = 0$ and we see that the minimal relation (3.7) actually involves only variables from $\mathbf{x}$ after all. Therefore the projection of H in E cannot be surjective.

(b) $\Rightarrow$ (a): Assume now that $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_j$ do not represent linearly independent classes in $H_{sr}(\Phi)$. Then by an automorphism of Q involving only a linear change of coordinates, we may assume that $\boldsymbol{\delta}_j$ is inner. Then, as we have seen above, we may conjugate by a matrix leaving the terms corresponding to $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_j - 1$ invariant to see that Q has a direct factor corresponding to $\boldsymbol{\delta}_j$. If there are further inner biderivations lying in $L\boldsymbol{\delta}_1 + \cdots + L\boldsymbol{\delta}_{j-1}$, we may proceed to find further direct factors of Q.

At any rate, in this case, the direct complement of these factors form a proper sub-t-module of Q which projects onto E. Thus Q is not a minimal extension of E in this case. □

Therefore we have the following result:

Corollary 3.4.2. *Let the L-linearly independent Φ-biderivations $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$ give rise to the quasi-periodic extension Q:*

$$0 \to \mathbb{G}_a^j \to Q \to E \to 0.$$

Then every maximal direct factor of Q lying in the kernel $\mathbb{G}_a^j$ has dimension equal to the dimension of $(\sum L\boldsymbol{\delta}_i) \cap \mathrm{Der}_{in}(\Phi)$

Proof. Assume that the indexing has been chosen so that $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_\ell$ are linearly independent in $H_{sr}(\Phi)$ and select an L-basis $\boldsymbol{\delta}'_{\ell+1}, \dots, \boldsymbol{\delta}'_j$ for $\mathrm{Der}_{in}(\Phi) \cap \sum L\boldsymbol{\delta}_i$. According to Lemma 3.3.3, Q is isomorphic, as an extension of E by $\mathbb{G}_a^j$ to $\widetilde{Q} = Q' \oplus \mathbb{G}_a^{j-\ell}$, where Q' is the extension of E arising from $\boldsymbol{\delta}'_1, \dots, \boldsymbol{\delta}_\ell$. According to Proposition 3.4.1, Q' is a minimal extension of E, so $\mathbb{G}_a^{j-\ell}$ is a maximal direct factor of Q lying in $\mathbb{G}_a^j$. This establishes one inequality.

On the other hand, suppose that Θ is an isomorphism of extensions of E by $\mathbb{G}_a^j$ so that

$$\Theta\Psi(t) = \begin{pmatrix} \Phi(t) & 0 \\ D(t) & \theta I_j \end{pmatrix} \Theta,$$

where $D(t) = (\boldsymbol{\delta}'(t), \dots, \boldsymbol{\delta}_{j'}(t), 0_d^t r, \dots, 0_d^{tr})^{tr}$. Write

$$\Theta = \begin{pmatrix} A_{d\times d} & B_{d\times j} \\ C_{j\times d} & D_{j\times j} \end{pmatrix},$$

where the index indicates the shape of the submatrix. Since Θ is an isomorphism of extensions of E by $\mathbb{G}_a^j$, $B_{d\times j}$ has all entries equal to zero and $D_{j\times j} \in \mathrm{GL}_{j\times j}(C_\infty\{\tau\})$. Then, since $D_{j\times j}\theta I_d = \theta D_{j\times j}$, $D_{j\times j} \in \mathrm{GL}_{j\times j}(C_\infty)$.

Thus we can interpret conjugation by Θ as a change of variables of $\sum L\boldsymbol{\delta}_i$ via $D_{j\times j}$, followed by the addition of inner biderivations via $C_{j\times d}$. Consequently we have that $j - j'$ linearly independent elements of $\sum L\boldsymbol{\delta}_i$ lie in $\mathrm{Der}_{in}(\Phi)$. Choosing j' maximal gives the other inequality. □

As a special case, we state the following corollary:

Corollary 3.4.3. *Let Δ be the space spanned over L by the Φ-biderivations $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$ and let Q be the corresponding quasi-periodic extension of E.*

(a) *Q is a split extension of E if and only if $\Delta \subset \mathrm{Der}_{in}(\Phi)$.*

(b) *Q is a minimal extension of E if and only if $0 = \Delta \cap \mathrm{Der}_{in}(\Phi)$*

3.5. **Cohomology under Isogeny of Abelian t-modules.** Here we extend a remark of [5] to arbitrary abelian t-modules. If Θ is a t-module morphism from $E_1 = (\Phi_1, \mathbb{G}_a^{d_1})$ to $E_2 = (\Phi_2, \mathbb{G}_a^{d_2})$, then it induces a C_∞-linear map $\Theta^* : \mathrm{Der}(\Phi_2) \to \mathrm{Der}(\Phi_1)$ via

$$(\Theta^*\boldsymbol{\delta})(a) := \boldsymbol{\delta}(a)\Theta, \ \forall a \in \mathsf{A}.$$

One sees from its functional equation that the quasi-periodic function associated to $\Theta^*\boldsymbol{\delta}$ is

$$F_{\Theta^*\boldsymbol{\delta}}(\mathbf{z}) = F_{\boldsymbol{\delta}}(d\Theta\mathbf{z}). \tag{3.11}$$

When we choose biderivations $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$ reducing to a basis for $H_{sr}(\Phi_1)$ and $j = r - d + \operatorname{rank} N$, write $\Theta^*\boldsymbol{\delta} = \boldsymbol{\delta}^{(\boldsymbol{U})} + \sum_{i=1}^{j} c_i\boldsymbol{\delta}_i$, then by Proposition 3.2.2 we can also write

$$F_{\Theta^*\boldsymbol{\delta}}(\mathbf{z}) = \boldsymbol{U}\,\mathrm{Exp}_{E_1}(\mathbf{z}) - d\boldsymbol{U}\mathbf{z} + \sum_{i=1}^{j} c_i F_{\boldsymbol{\delta}_i}(\mathbf{z}). \tag{3.12}$$

Proposition 3.5.1. *If Θ is an isogeny from the abelian t-module $E_1 = (\Phi_1, \mathbb{G}_a^d)$ to $E_2 = (\Phi_2, \mathbb{G}_a^d)$, then Θ^* induces isomorphisms*

(a) $\Theta^*_{\mathrm{DR}} : H_{\mathrm{DR}}(\Phi_2) \to H_{\mathrm{DR}}(\Phi_1)$ *and*
(b) $\Theta^*_{sr} : H_{sr}(\Phi_2) \to H_{sr}(\Phi_1)$.

Proof. By isogeny E_1 and E_2 have the same dimension and rank; furthermore, $N_1 = d\Phi_1(t) - \theta I_d$ and $N_2 = d\Phi_2(t) - \theta I_d$ have the same rank because $d\Theta N_1 = N_2 d\Theta$. In light of the dimensions calculated in Proposition 3.1.3, it suffices to show that Θ^*_{DR} and Θ^*_{sr} are injective.

Since $\Theta : E_1 \to E_2$ is an isogeny, there is an isogeny $\Omega : E_2 \to E_1$ and a non-zero element $a \in \mathsf{A}$ so that

$$\Omega\Theta = \Phi_1(a).$$

Clearly $(\Omega\Theta)^* = \Theta^*\Omega^* = \Phi_1(a)^*$, and so we need only show that $\Phi_1(a)^*_{\mathrm{DR}}$ and $\Phi_1(a)^*_{sr}$ are injective.

Suppose that $\Phi_1(a)^*\boldsymbol{\delta}$ is inner (or strictly inner). Then

$$(\Phi_1(a)^*\boldsymbol{\delta})(b) = \boldsymbol{\delta}(b)\Phi_1(a) = \boldsymbol{U}\Phi_1(b) - \iota(b)\boldsymbol{U}$$

for some $\boldsymbol{U} \in M(E_1)$ with $d\boldsymbol{U} \in N_1^{\perp}$ (or $d\boldsymbol{U} = 0$) and for all $b \in \mathsf{A}$.

Let $\boldsymbol{V} := (\iota(a))^{-1}(\boldsymbol{U} - \boldsymbol{\delta}(a))$. Since $\boldsymbol{V}\Phi_1(a) = \boldsymbol{U}$, we see that

$$\boldsymbol{\delta}^{(\boldsymbol{V})}(b)\Phi_1(a) = \boldsymbol{V}\Phi_1(ba) - \iota(b)\boldsymbol{V}\Phi_1(a) = \boldsymbol{U}\Phi_1(b) - \iota(b)\boldsymbol{U} = \boldsymbol{\delta}(b)\Phi_1(a).$$

Since $M(E_1)$ has no t-torsion, we can cancel the common right factor to see that $\boldsymbol{\delta} = \boldsymbol{\delta}^{(\boldsymbol{V})}$, which is strictly inner if and only if $\Phi_1(a)^*\boldsymbol{\delta}$ is so. □

Corollary 3.5.2. *Let Θ be an isogeny from the abelian t-module $E_1 = (\Phi_1, \mathbb{G}_a^d)$ to $E_2 = (\Phi_2, \mathbb{G}_a^d)$. Let $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$ be representatives of linearly independent classes of $H_{sr}(\Phi_2)$. Let Q_2 be the quasi-periodic extension of E_2 associated to $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$, and let Q_1 be the quasi-periodic extension of E_1 associated to $\Theta^*\boldsymbol{\delta}_1, \dots, \Theta^*\boldsymbol{\delta}_j$. Then the matrix*

$$\Theta_* := \begin{pmatrix} \Theta & 0 \\ 0 & I_j \end{pmatrix},$$

where I_j is the $j \times j$ identity matrix, is an isogeny from Q_1 to Q_2.

Proof. According to the preceding proposition, the $\Theta^*\boldsymbol{\delta}_i$ are representatives of a basis for $\mathrm{Der}(\Phi_1)/\mathrm{Der}_{in}(\Phi_1)$. Since

$$\begin{pmatrix} \Theta & 0 \\ 0 & I_j \end{pmatrix} \begin{pmatrix} \Phi_1(t) & 0 & 0 & \dots & 0 \\ \Theta^*\boldsymbol{\delta}_1(t) & \theta & 0 & \dots & 0 \\ \Theta^*\boldsymbol{\delta}_2(t) & 0 & \theta & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \Theta^*\boldsymbol{\delta}_j(t) & 0 & 0 & \dots & \theta \end{pmatrix} = \begin{pmatrix} \Phi_2(t) & 0 & 0 & \dots & 0 \\ \boldsymbol{\delta}_1(t) & \theta & 0 & \dots & 0 \\ \boldsymbol{\delta}_2(t) & 0 & \theta & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \boldsymbol{\delta}_j(t) & 0 & 0 & \dots & \theta \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & I_j \end{pmatrix},$$

we see that Θ_* is an isogeny from Q_1 to Q_2. □

Corollary 3.5.3. *Let Q_i be strictly quasi-periodic extensions of the abelian t-modules E_i of maximal dimension, $i = 1, \ldots, m$. Let E be isogenous to $\prod E_i$. Then any strictly quasi-periodic extension Q of the t-module E of maximal dimension is isogenous to $\prod Q_i$.*

Proof. Let Q_i arise from the Φ_i-biderivations $\boldsymbol{\delta}_{i,j}, j = 1, \ldots, j_i$. Then according to Part (b) of Proposition 3.5.1, the $\Theta^* \boldsymbol{\delta}_{i,j}$ are linearly independent modulo $\mathrm{Der}_{in}(\Phi)$.

As in the proof of part (a) of Proposition 3.5.1,

$$\operatorname{rank}\left(d\Phi(t) - \theta I_d\right) = \sum \operatorname{rank}\left(d\Phi_i(t) - \theta I_{d_i}\right).$$

Moreover, $\operatorname{rank} E = \sum \operatorname{rank} E_i, \dim E = \sum \dim E_i$. Thus, according to Lemma 3.1.1,

$$\begin{aligned} \dim_L H_{sr}(\Phi) &= \operatorname{rank} E - \dim E + \operatorname{rank}(d\Phi(t) - \theta I_d) \\ &= \sum_i \left(\operatorname{rank} E_i - \dim E_i + \operatorname{rank}(d\Phi_i(t) - \theta I_{d_i})\right) \\ &= \sum j_i. \end{aligned}$$

Consequently, the $\Theta^* \boldsymbol{\delta}_{i,j}$ span $H_{sr}(\Phi)$, and we conclude via Proposition 3.3.2. □

Remark. Let $\Theta : E_1 \to E_2$ be an isogeny between simple t-modules defined over L. Then Θ is defined over L, as L is algebraically closed.

Corollary 3.5.4. *If E_1 and E_2 are isogenous t-modules defined over L, then the L-vector space spanned by the coordinates of the periods of E_i is independent of i. The same is true of the L-vector space spanned by the quasi-periods and the coordinates of the periods for maximal strictly quasi-periodic extensions defined over L.*

Proof. Let $\Theta : E_1 \to E_2$ be an isogeny and let Λ_1, Λ_2 denote the period lattices for E_1, E_2. Then $d\Theta\Lambda_1$ is an A-sublattice of Λ_2 of finite index. Thus the L-spans of the lattices are the same.

We saw in the preceding result that such an isogeny Θ induces an isogeny Θ_* of the related minimal quasi-periodic extensions. The claim follows on appealing to (3.12). □

4. t-modules Arising from Solitons

In [3], Anderson introduced the notion of a soliton function, which is a higher dimensional analogue of the shtuka function for rank 1 Drinfeld modules. Coleman had given such meromorphic functions explicitly for the Fermat and Artin-Schreier curves [7], and Thakur explained these examples in terms of the Gamma function. Anderson observed the parallel between these ideas and certain topics in differential equations and thus developed "solitons."

Sinha used the soliton theory in his Minnesota Ph.D. thesis [15, 16, 17] to construct t-modules whose periods have coordinates which are algebraic multiples of $\Gamma(a/f)$ with a and f in A both monic with $\deg(a) < \deg(f)$.

In this section we propose to construct t-modules whose periods involve arbitrary $\Gamma(a/f)$ with $\deg(a) < \deg(f)$, and to determine their sub-t-module structure in terms of Thakur's bracket relations.

In order to render a coherent account here of our own considerations, we have found it necessary to first recapitulate in Sections 4.1–4.3, those concepts, constructions, and conclusions of Sinha to which we appeal. We believe that in this way, the reader may discern the flow of the arguments much better. The expert may simply glance through this section to reassure himself or herself of the notation we have adopted. The newcomer may well find it a guide for the exploration of the related aspects of Sinha's far-ranging article.

4.1. **Soliton Functions.** We fix some notation. In all that follows, unless otherwise specified, we will take fiber products and tensor products over $\mathbb{F}_q$: i.e. $\times := \times_{\mathbb{F}_q}$ and $\otimes := \otimes_{\mathbb{F}_q}$.

Recall that the Carlitz module $(C, \mathbb{G}_a)$ is the 1-dimensional t-module defined by

$$C(t) = \theta + \tau,$$

with exponential function $e_C : C_\infty \to C_\infty$. Fix throughout $f \in A_+ = \{a \in A : a \text{ is monic}\}$. Let

$$\zeta_f := e_C\left(\frac{\widetilde{\pi}}{f}\right),$$

and let

$$B := B_f := \mathbb{F}_q[\theta, \zeta_f]$$

be the integral closure of A in $K := K_f := k(\zeta_f)$. The group $(A/f)^\times$ is isomorphic to the Galois group $\mathrm{Gal}(K/k)$ via

$$\sigma_a e_C\left(\frac{b\widetilde{\pi}}{f}\right) = e_C\left(\frac{ab\widetilde{\pi}}{f}\right). \tag{4.1}$$

The infinite place ∞ of k is ramified in K, with inertia and decomposition group the natural subgroup $\mathbb{F}_q^\times \subset (A/f)^\times$. Thus if we take $K_+ \subset K$ to be the maximal subfield in which ∞ is totally split, then

$$[K : K_+] = q - 1,$$

and

$$[K_+ : k] = \big|\{a \in A_+ : \deg(a) < \deg(f),\, (a, f) = 1\}\big|.$$

We take B_+ for the ring of integers of K_+. Moreover, if we let

$$\mathcal{I}_+ = \{a \in A_+ : \deg(a) < \deg(f),\, (a, f) = 1\}, \tag{4.2}$$

and

$$\mathcal{I} = \{a \in A : \deg(a) < \deg(f),\, (a, f) = 1\}, \tag{4.3}$$

then $\mathrm{Gal}(K_f/k) = \{\sigma_a : a \in \mathcal{I}\}$ and $\mathrm{Gal}(K_+/k) = \{\sigma_a|_{K_+} : a \in \mathcal{I}_+\}$. We outline Sinha's definition of an important class of t-modules $E_f := (\Phi_f, \mathbb{G}_a^d)$ of dimension $d := |\mathcal{I}_+|$ using Anderson's soliton function as follows.

Let $X/\mathbb{F}_q$ be an irreducible smooth projective curve with function field K, i.e. we choose an isomorphism $\xi : \mathbb{F}_q(X) \to K$, which amounts to providing a K-valued point $\xi \in X(K)$. Let $U := \mathrm{Spec}(B) \subset X$, and take $\overline{\infty}$ for the complement of U, i.e. the points extending ∞ in the natural map $X \to \mathbb{P}^1$. Let $V \subset X$ be the open subscheme which is the complement of the zeros of f. For $a \in (A/f)^\times$ we take $[a] : X \to X$ over $\mathbb{F}_q$ for the automorphism induced by the morphism $\sigma_a^{-1} : B \to B$ of A-algebras (going in the opposite direction).

Theorem 4.1.1 (Anderson [3])**.** *Let $f \in A_+$. There exists a unique rational function ϕ on $X \times X$, regular on $V \times U$, such that for all $a \in A$ with $\deg(a) < \deg(f)$ and $(a, f) = 1$ and for all positive integers N,*

$$1 - \phi(\mathrm{Frob}^N(\xi), [a]\xi) = \prod_{\substack{n \in A_+ \\ \deg(n) = N-1}} \left(1 + \frac{a}{fn}\right)$$

where $\mathrm{Frob} : X \to X$ *is the q-th power Frobenius morphism.*

In fact Anderson and Sinha completely determine the divisors of ϕ and $1 - \phi$ on $X \times X$, see [3, 15] for details. These divisor identities led Anderson to term ϕ a *soliton function for X.*

We will not use the full two-variable version of soliton functions, but rather a one-variable version as follows. We extend scalars on X by an algebraically closed field L/k (e.g., $L = \overline{k}$ or $L = C_\infty$). We then obtain the curve over L

$$\mathbf{X} := \operatorname{Spec} L \times X.$$

We note that there are copies of the rings, A, B, K, etc., contained in the function field of $\mathbf{X}$ once as *functions on* $\mathbf{X}$ and once as *scalars.* To make the distinction between the two interpretations, we take $\theta \in L$ for the constant function on $\mathbf{X}$ and $t \in L(\mathbf{X})$ for the function on $\mathbf{X}$ such that $t(\xi) = \theta$. On $\mathbf{X}$, the function $t - \theta$ then has divisor

$$\operatorname{div}(t - \theta) = \left(\sum_{a \in \mathcal{I}} [a]\xi\right) - (q-1)I, \tag{4.4}$$

where

$$I := \operatorname{Spec} L \times \overline{\infty}.$$

When appropriate, we will use the notation $\mathsf{A} = \mathbb{F}_q[t]$, B, K, etc., for copies of A, B and K with underlying variable $t \in \mathsf{A}$ playing the role of $\theta \in A$. As before, define $\iota : t \mapsto \theta$ to be the corresponding isomorphism.

Via the action of the q-th power Frobenius $\tau : L \to L$, it is possible to conjugate functions, divisors, etc. on $\mathbf{X}$. For a rational (or analytic) function r on $\mathbf{X}$, we let $r^{(1)}$, $r^{(2)}$, ..., denote successive conjugations. Note that $r^{(1)}$ is obtained by raising the coefficients of r to the q-th power. Likewise, for a divisor D on $\mathbf{X}$, the conjugate $D^{(1)}$ is obtained by raising the coordinates of the points in the support of D (in some coordinate system) to the q-th power.

Definition. We can also pull-back functions on $X \times X$ by the natural map

$$\mathbf{X} = \operatorname{Spec} L \times X \to X \times X.$$

Given $f \in \mathsf{A}_+$, the single-variable *Anderson-Coleman soliton function g_f,* or simply *soliton function,* is defined to be the rational function in $L(\mathbf{X})$,

$$g := g_f := 1 - \phi_f : \mathbf{X} \to L.$$

Anderson's theorem now leads to the following result, which provides the connection between solitons and Gamma values at rational arguments:

Proposition 4.1.2 (Anderson [3])**.** *For all $a \in \mathcal{I}$ and $N > 0$,*

$$g_f^{(N)}([a]\xi) = \prod_{\substack{n \in A_+ \\ \deg(n) = N-1}} \left(1 + \frac{a}{fn}\right).$$

Note that the right-hand side in the above equation is the reciprocal of a partial product of $\Gamma(a/f)$.

4.2. **Soliton t-modules.**

4.2.1. *Divisors of Solitons.* Fix $f \in \mathsf{A}_+$. We define the following effective divisors on $\mathbf{X}$:

$$(4.5)\qquad \begin{aligned} W := W_f &:= \sum_{j=0}^{\deg(f)-2} \sum_{\substack{a \in \mathcal{I}_+ \\ \deg(a) \le j}} [a] \circ \mathrm{Frob}^{\deg(f)-j-2}(\xi), \\ \Xi := \Xi_f &:= \sum_{a \in \mathcal{I}_+} [a]\xi. \end{aligned}$$

The divisor of the soliton function g is then shown to be

$$(4.6)\qquad \mathrm{div}(g) = W^{(1)} - W + \Xi - I.$$

Moreover, Sinha [16] uses this fact to construct t-modules in the following manner.

4.2.2. *Soliton t-motives.* Recall that $U \subset X$ is the open set isomorphic to $\operatorname{Spec} B$. We take $\mathbf{U} \subset \mathbf{X}$ to be

$$\mathbf{U} := \operatorname{Spec} L \times U.$$

Let $\Omega_{\mathbf{X}}$ be the sheaf of meromorphic 1-forms on $\mathbf{X}$, and define

$$M_f := \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W))$$

to be the usual L-vector space of 1-forms which are regular on $\mathbf{U}$ with at worst poles (all simple) along W. As defined, M_f is a $L[t]$-module under left-multiplication, since t is a rational function on $\mathbf{X}$ which is regular on $\mathbf{U}$. In analogy with Drinfeld's shtuka approach to Drinfeld modules, M_f is made into a $L\{\tau\}$-module by defining

$$(4.7)\qquad \tau \cdot m := g_f m^{(1)},$$

for every $m \in M_f$. Sinha then proves the following theorem.

Theorem 4.2.1 (Sinha [16, §3 and §5])**.** *With the given $L[t, \tau]$-module structure M_f is a uniformizable abelian t-motive defined over L.*

Under the equivalence of categories between t-motives and t-modules, we take

$$E_f := E(M_f),$$

to be the abelian t-module associated to M_f (see Section 2.1). Sinha then computes the dimension and rank of E_f as a t-module to be:

$$(4.8)\qquad d := \dim(E_f) := \operatorname{rank}_{L\{\tau\}} M_f = |\mathcal{I}_+|,$$

and

$$(4.9)\qquad r := \operatorname{rank}(E_f) := \operatorname{rank}_{L[t]} M_f = |\mathcal{I}|,$$

where $\mathcal{I}_+$ and $\mathcal{I}$ are defined as in (4.2) and (4.3). We put $E_f = (\Phi_f, \mathbb{G}_a^d)$, i.e we take $\Phi := \Phi_f$ for the homomorphism defining the t-module action on E_f.

Remark. The dimension of E_f is equal to $[\mathsf{K}_+ : \mathsf{k}]$, where K_+ is an extension of k which is totally split at ∞, i.e. the maximal real subfield of K_f. Furthermore, the elements of B_f are rational functions on $\mathbf{X}$ which are regular on $\mathbf{U}$, so M_f is a B_f-module. In Section 4.3 (see Remark 4.3.2) we will see how the action of $d\Phi(\mathsf{B}_f)$ on $\mathrm{Lie}(E_f)$ extends the conjugate action of B_+ as in Sections 2.2–2.3. Thus E_f will be seen to be a Hilbert-Blumenthal-Drinfeld module with real multiplications by B_+ plus complex multiplications by all of B_f, making it a t-module of CM-type.

4.2.3. *Exponential Functions of Soliton t-modules.* We proceed as in Sinha [16, §3–4]. For the rest of this subsection we take $L = C_\infty$. We begin with a discussion on analytic functions on $\mathbf{X}$. First, as $C_\infty(t) = C_\infty \otimes \mathsf{k}$ is the function field of $\mathbb{P}^1/C_\infty$, we interpret the *Tate algebra*,

$$\mathfrak{A} := C_\infty\{t/\theta\} := \left\{ \sum_{i=0}^{\infty} a_i t^i \in C_\infty[\![t]\!] \;\middle|\; \lim_{i\to\infty} |\theta^i a_i| = 0 \right\},$$

as the C_∞-algebra of functions on $\mathbb{P}^1$ which are analytic in the closed disk of radius $|\theta|$ centered at 0, i.e. the rigid analytic disk of radius $|\theta|$. Extend $\mathfrak{A}$ to

$$\mathfrak{B} := \mathsf{B}_f \otimes_{\mathsf{A}} \mathfrak{A} = \mathfrak{A}[\zeta_f].$$

Letting $\mathfrak{X}$ be the rigid analytic variety associated to $\mathbf{X}$, then $\mathfrak{B}$ is the C_∞-algebra of functions on $\mathfrak{X}$ which are analytic on $\mathfrak{U}$, defined to be the inverse image under t of the closed disk of radius $|\theta|$ about 0. In particular $\mathfrak{U}$ contains the zeros of $t - \theta$,

$$[\mathcal{I}]\xi := \sum_{a \in \mathcal{I}} [a]\xi,$$

as in (4.4). Moreover, $\mathfrak{U}$ is the affinoid variety contained in $\mathfrak{X}$ associated to $\mathfrak{B}$. Note that as the C_∞-valued points of $\mathbf{X}$ and $\mathfrak{X}$ exactly coincide, we are free to consider points and divisors on $\mathbf{X}$ as points and divisors on $\mathfrak{X}$.

Following Sinha, we let

$$\mathcal{R} := \frac{\left\{ \alpha \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}(-W + \Xi)) \;\middle|\; \alpha - \dfrac{\alpha^{(1)}}{g} \in \Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W + \Xi)) \right\}}{\Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W))}. \tag{4.10}$$

In light of the discussion in the previous paragraph, modulo those rational functions on $\mathbf{X}$ which vanish along W, elements of $\mathcal{R}$ are analytic functions on $\mathfrak{U}$ which have zeros along W and possibly poles (all simple) along Ξ and which, under the operation

$$\alpha \mapsto \alpha - \frac{\alpha^{(1)}}{g}, \tag{4.11}$$

become rational functions on $\mathbf{X}$ (which themselves vanish along W and might have poles along Ξ). The operation in (4.11) allows us to continue $\alpha \in \mathcal{R}$ meromorphically to all of $\mathbf{U}$, with poles possibly at points of $\Xi^{(i)}$ for $i \geq 0$.

Sinha then obtains the exponential function of E_f by constructing a commutative diagram via residues

$$\begin{array}{ccccc} & & \mathcal{R} & & \\ & \swarrow^{\mathcal{R}\mathrm{Lie}=\mathrm{res}_{\Xi}} & & \searrow^{\mathcal{R}\mathrm{Exp}=\mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}} & \\ \mathrm{Lie}(E_f) & & \xrightarrow{\ \mathrm{Exp}\ } & & E_f(C_\infty), \end{array} \tag{4.12}$$

where the map $\mathcal{R}\mathrm{Lie} : \mathcal{R} \to \mathrm{Lie}(E_f)$ is an isomorphism of $\mathbb{F}_q$-vector spaces.

To define $\mathcal{R}\mathrm{Lie}$ and $\mathcal{R}\mathrm{Exp}$, we first choose a basis $\{n_1, \ldots, n_d\}$ for M_f as a $C_\infty\{\tau\}$-module. Since M_f is defined to be $\Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W))$, it follows that for any $\alpha \in \mathcal{R}$, the differential αn_j will be regular on $\mathbf{U}$ except for possible poles along $\Xi^{(i)}$, $i \geq 0$. The map

$$\mathcal{R}\mathrm{Lie} : \mathcal{R} \to \mathrm{Lie}(E_f) \simeq C_\infty^d$$

is defined by

$$\mathcal{R}\mathrm{Lie} : \alpha \mapsto \begin{pmatrix} \mathrm{res}_{\Xi}(\alpha n_1) \\ \vdots \\ \mathrm{res}_{\Xi}(\alpha n_d) \end{pmatrix}, \tag{4.13}$$

where res_{Ξ} represents the sum of residues over all points of Ξ. Sinha shows that (4.13) is an isomorphism of $\mathbb{F}_q$-vector spaces via the calculation of certain Ext groups (see [16, §4]). The exponential function is then obtained from the map

$$\mathcal{R}\mathrm{Exp} : \mathcal{R} \to E_f(C_\infty)$$

defined by

$$\mathcal{R}\mathrm{Exp} : \alpha \mapsto \begin{pmatrix} \sum_{i=0}^{\infty} \mathrm{res}_{\Xi^{(i)}}(\alpha n_1) \\ \vdots \\ \sum_{i=0}^{\infty} \mathrm{res}_{\Xi^{(i)}}(\alpha n_d) \end{pmatrix}, \tag{4.14}$$

so that $\mathrm{Exp}_{E_f}(\mathbf{z})$ is obtained by composition of $\mathcal{R}\mathrm{Exp}$ with the inverse of $\mathcal{R}\mathrm{Lie}$.

Remark 4.2.2. Because we will need the technique later, we demonstrate that the map $\mathcal{R}\mathrm{Exp}$ satisfies the functional equation determined by the t-module structure on E_f. It can further be shown that map $\mathcal{R}\mathrm{Exp} \circ \mathcal{R}\mathrm{Lie}^{-1}$ is analytic and that its coordinate functions are normalized as in Section 2.1.2 (cf. Proposition 4.4.1). For our given $C_\infty\{\tau\}$-basis for M_f, we have a representation

$$t n_i = \sum_j \Phi(t)_{ij} n_j,$$

where $\Phi_f(t) = (\Phi(t)_{ij}) \in \mathrm{Mat}_{d\times d}(C_\infty\{\tau\})$ is the multiplication-by-t action on E_f as in (2.1). Therefore

$$\mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(t\alpha n_i) = \sum_j \mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha \Phi(t)_{ij} n_j).$$

To verify that $\mathcal{R}\mathrm{Exp}$ satisfies the functional equation, it is enough to show for every $m \in M_f$ that

$$\mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha\tau m) = (\mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha m))^q.$$

Recall from the definition of α in (4.10) that $g\alpha - \alpha^{(1)} = s \in \Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W^{(1)}))$; thus $sm^{(1)}$ has no poles on $\mathbf{U}$. Therefore, from (4.7)

$$\begin{aligned} \mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha\tau m) &= \mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha g m^{(1)}) \\ &= \mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha^{(1)} m^{(1)}) + \mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(s m^{(1)}) \\ &= (\mathrm{res}_{\Xi^{(0)}+\Xi^{(1)}+\cdots}(\alpha m))^q. \end{aligned} \tag{4.15}$$

Remark 4.2.3. Again because we will need the technique below, we record here the fact that the map $\mathcal{R}\mathrm{Exp} : \mathcal{R} \to E_f(C_\infty)$ can also be defined by taking residues along $I = \mathrm{Spec}\, L \times \overline{\infty}$: For $\alpha \in \mathcal{R}$, let $r := \alpha - \alpha^{(1)}/g \in \Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W + \Xi))$.

$$\mathcal{R}\mathrm{Exp} : \alpha \mapsto - \begin{pmatrix} \mathrm{res}_I \left(\left(r + \frac{r^{(1)}}{g^{(0)}} + \frac{r^{(2)}}{g^{(0)}g^{(1)}} + \cdots \right) n_1 \right) \\ \vdots \\ \mathrm{res}_I \left(\left(r + \frac{r^{(1)}}{g^{(0)}} + \frac{r^{(2)}}{g^{(0)}g^{(1)}} + \cdots \right) n_d \right) \end{pmatrix}.$$

This representation relies on the following lemma (see [8, §2] or [9, §I.3.3]).

Lemma 4.2.4. *Let β be a meromorphic function on $\mathbf{U}$ with discrete poles, and let n be an algebraic differential form on $\mathbf{X}$. Let m_j be a strictly increasing sequence of integers such that $\omega := \beta n$ is regular on $\{\gamma \in \mathbf{U} : |t(\gamma)| = |\theta|^{m_j}\}$. If for every real number $\rho > 0$, $\lim_{j\to\infty} \rho^{m_j} \sup_{|t(\gamma)|=|\theta|^{m_j}} \{|\beta|\} = 0$, then*

$$\sum_{\gamma\in\mathbf{U}} \mathrm{res}_\gamma(\omega) = 0.$$

Combining the identity

$$\alpha - \frac{\alpha^{(N+1)}}{g^{(0)} \cdots g^{(N)}} = r + \frac{r^{(1)}}{g} + \cdots + \frac{r^{(N)}}{g \cdots g^{(N-1)}}, \tag{4.16}$$

which is established recursively, we find

$$\begin{aligned} \sum_{i=0}^{\infty} \mathrm{res}_{\Xi^{(i)}}(\alpha n_j) = \sum_{i=0}^{\infty} &\mathrm{res}_{\Xi^{(i)}} \left(\frac{\alpha^{(N+1)} n_j}{g^{(0)} \cdots g^{(N)}} \right) \\ &- \mathrm{res}_I \left(\frac{r^{(N)} n_j}{g^{(0)} \cdots g^{(N-1)}} \right) - \cdots - \mathrm{res}_I(r n_j). \end{aligned}$$

Sinha [16, Lemma 4.6.4] shows that for any real number $\rho > 0$, we have

$$\lim_{N\to\infty} \rho^{q^N} \sup_{|t(\gamma)|=|\theta|^{q^N - q^{N-2}}} \left\{ |\alpha^{(N)} n_j/(g^{(0)} \cdots g^{(N-1)})| \right\} = 0.$$

Combining this with Lemma 4.2.4, Sinha determines for N sufficiently large that

$$\sum_{\gamma\in\mathbf{U}} \mathrm{res}_\gamma \left(\frac{\alpha^{(N+1)} n_j}{g^{(0)} \cdots g^{(N)}} \right) = \sum_{i=0}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{\alpha^{(N+1)} n_j}{g^{(0)} \cdots g^{(N)}} \right) = 0, \tag{4.17}$$

since on $\mathbf{U}$ the poles of $\alpha^{(N)} n_j/(g^{(0)} \cdots g^{(N-1)})$ lie along $\Xi^{(0)} + \Xi^{(1)} + \cdots$.

4.3. **Periods of Soliton t-modules.** In the next important step, Sinha determines the kernel of the map $\mathcal{R}\mathrm{Exp} : \mathcal{R} \to E_f(C_\infty)$ from (4.12), and *a fortiori* obtains the period lattice of E_f.

Consider the infinite product

$$c_g = \frac{1}{g^{(0)}g^{(1)}\dots},$$

which converges in the space of rigid analytic maps $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}(-W + \Xi))$. Clearly,

$$c_g - \frac{c_g^{(1)}}{g} = 0,$$

so $c_g \in \mathcal{R}$. From Remark 4.2.3 it then follows that c_g is in the kernel of the exponential map $\mathcal{R}\mathrm{Exp}$, and in fact Sinha shows

$$\mathrm{Ker}(\mathcal{R}\mathrm{Exp}) = B_f \cdot c_g.$$

The period lattice Λ_f of E_f is then the image of $\mathrm{Ker}(\mathcal{R}\mathrm{Exp})$ under $\mathcal{R}\mathrm{Lie} : \mathcal{R} \to \mathrm{Lie}(E_f)$. Thus Λ_f is an A-module of rank $[\mathsf{K}_f : \mathsf{k}]$, since it is a free B_f-module of rank 1.

Following Sinha, we now choose a $C_\infty\{\tau\}$-basis for M_f which is amenable to computing the period lattice explicitly. Define

$$\varepsilon(a) = \begin{cases} 1, & \text{if } \deg(a) = \deg(f) - 1, \\ 0, & \text{if } \deg(a) < \deg(f) - 1. \end{cases}$$

Note for $a \in \mathcal{I}$ that $\varepsilon(a)$ is the multiplicity of $[a]\xi$ in $\Xi - W$. Sinha then chooses a $C_\infty\{\tau\}$-basis $\{n_a\}_{a\in\mathcal{I}_+}$ for M_f so that n_a is defined over $\overline{k}$;

$$n_a \in \Gamma(\mathbf{U}, W - \Xi + [a]\xi);$$
$$\mathrm{res}_{[a]\xi}((n_a)/(t-\theta)^{\varepsilon(a)}) = 1;$$

and

$$n_a \text{ has } \begin{cases} \left.\begin{array}{l} \text{a pole of order 1 at } [a]\xi, \\ \text{no pole at } [b]\xi \text{ if } b \neq a,\ \varepsilon(b) = 0, \\ \text{a zero at } [b]\xi \text{ if } \varepsilon(b) = 1, \end{array}\right\} & \text{if } \varepsilon(a) = 0; \\ \left.\begin{array}{l} \text{no pole or zero at } [a]\xi, \\ \text{no pole at } [b]\xi \text{ if } \varepsilon(b) = 0, \\ \text{a zero at } [b]\xi \text{ if } b \neq a,\ \varepsilon(b) = 1, \end{array}\right\} & \text{if } \varepsilon(a) = 1. \end{cases}$$

Thus determining Λ_f amounts to computing

$$\pi_a := \mathrm{res}_{\Xi}(c_g n_a) = \mathrm{res}_{[a]\xi}(c_g n_a),$$

since the image of arbitrary $b \cdot c_g$ in $\Lambda_f \subset \mathrm{Lie}(E_f)$ is then

$$\mathcal{R}\mathrm{Exp}(c_g \cdot b) = \begin{pmatrix} \vdots \\ \sigma_a(\iota(b))\pi_a \\ \vdots \end{pmatrix}_{a\in\mathcal{I}_+}. \tag{4.18}$$

The function b is evaluated at the point $[a]\xi$, for which we see $b([a]\xi) = \sigma_a(\iota(b))$. Note that this shows that the action of $d\Phi(\mathsf{B}_+)$ on $\mathrm{Lie}(E_f)$ is the conjugate action $\boldsymbol{\sigma}(\mathsf{B}_+)$ of K_+/k, where $\boldsymbol{\sigma} = \oplus\sigma_a|_{\mathsf{K}_+}$ for $a \in \mathcal{I}_+$.

We now compute $\mathrm{res}_{[a]\xi}(c_g n_a)$. From (4.5) and (4.6) it follows that the function $c_g = (g^{(0)}g^{(1)}\cdots)^{-1}$ has poles among the points $[a]\xi$ of Ξ exactly when $\varepsilon(a)=1$. Thus if $\varepsilon(a)=0$, then by Proposition 4.1.2,

$$\begin{aligned}\pi_a = \mathrm{res}_{[a]\xi}(c_g n_a) &= c_g([a]\xi) \\ &= g([a]\xi)^{-1}\prod_{n\in A_+}\left(1+\frac{a}{fn}\right)^{-1} \\ &= g([a]\xi)^{-1}\frac{a}{f}\Gamma\left(\frac{a}{f}\right).\end{aligned} \tag{4.19}$$

Likewise, if $\varepsilon(a)=1$, then by a similar calculation

$$\pi_a = \mathrm{res}_{[a]\xi}(c_g n_a) = \mathrm{res}_{[a]\xi}(n_a/g)\frac{a}{f}\Gamma\left(\frac{a}{f}\right).$$

We see right away for all $a\in\mathcal{I}_+$ that

$$\pi_a \sim \Gamma\left(\frac{a}{f}\right), \tag{4.20}$$

because n_a, g, and ξ are defined over $\overline{k}$. Sinha further computes these algebraic factors and obtains the following explicit result after some careful analysis.

Theorem 4.3.1 (Sinha [16, §5.3.9])**.** *Let a, $f\in A$ be monic with* $\deg(a)<\deg(f)$ *and* $(a,f)=1$*. Then*

$$\pi_a = \begin{cases}\Gamma\left(\dfrac{a}{f}\right), & \text{if } \deg(a)=\deg(f)-1,\\[2ex] \dfrac{a}{f}\Gamma\left(\dfrac{a}{f}\right), & \text{if } \deg(a)<\deg(f)-1.\end{cases}$$

Remark 4.3.2. From the preceding choice of coordinates on $\mathrm{Lie}(E_f)$, in particular (4.18), we see that E_f is a H-B-D module with real multiplications by B_+. Furthermore, if we let

$$\mathcal{S}_f = \{\sigma_a \in \mathrm{Gal}(\mathsf{K}_f/\mathsf{k}) : a\in\mathcal{I}_+\},$$

then $\mathcal{S}_f$ is an extension of $\mathrm{Gal}(\mathsf{K}_+/\mathsf{k})$ to $\mathrm{Gal}(\mathsf{K}_f/\mathsf{k})$. From (4.18) we see that E_f is a t-module of CM-type $(\mathsf{K}_f,\mathcal{S}_f)$ with complex multiplications by B_f and conjugate action $\boldsymbol{\sigma}_f := \boldsymbol{\sigma}_{\mathcal{S}_f}$.

4.4. **Quasi-periodic Soliton t-modules and Quasi-periods.** We now apply the ideas underlying Sinha's construction to the quasi-periodic extensions of soliton t-modules defined Section 3. Since $d\Phi_f(t)$ for the soliton t-module E_f has no nilpotent part, i.e. $(t-\theta)M_f \subset \tau M_f$, we see from Proposition 3.1.3 that

$$\dim_{C_\infty}\mathrm{Der}_0(\Phi_f) = d \quad\text{and}\quad \dim_{C_\infty} H_{sr}(\Phi_f) = r-d,$$

where d and r are the dimension and rank of E_f defined in (4.8) and (4.9). Moreover, we know that

$$H_{sr}(\Phi_f) \simeq \frac{\tau M_f}{(t-\theta)M_f}, \tag{4.21}$$

and our immediate task is to find a convenient basis for this space (defined over $\overline{k}$) and compute the associated quasi-periodic functions and quasi-periods.

4.4.1. *Quasi-periodic Functions.* As in Section 3.1, given an element of the t-motive $m \in \tau M_f$, we have an associated biderivation $\boldsymbol{\delta} := \boldsymbol{\delta}_m : \mathsf{A} \to (C_\infty\{\tau\}\tau)^d$ defined by

$$\boldsymbol{\delta}(t) = (\delta_1(t), \ldots, \delta_d(t)),$$

where $m = \sum \delta_j(t) n_j$ for a fixed $C_\infty\{\tau\}$-basis $\{n_1, \ldots, n_d\}$ of M_f.

We now proceed to express the associated quasi-period function $F_{\boldsymbol{\delta}} : \mathrm{Lie}(E_f) \to C_\infty$ in terms of residues, as in (4.12). Consider the map $\mathcal{R}F_{\boldsymbol{\delta}} : \mathcal{R} \to C_\infty$ defined by

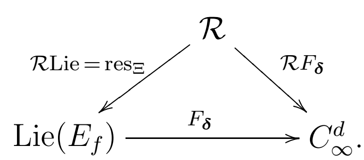
$$\mathcal{R}F_{\boldsymbol{\delta}} : \alpha \mapsto \sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{\alpha m}{t-\theta} \right).$$

If $s \in \Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W))$, then $sm/(t-\theta) \in \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(-[\mathcal{I}]\xi - \Xi))$. Thus $\mathcal{R}F_{\boldsymbol{\delta}}(s) = 0$, and so $\mathcal{R}F_{\boldsymbol{\delta}}$ is well-defined. Moreover, $\mathcal{R}F_{\boldsymbol{\delta}}$ essentially gives the quasi-periodic function for $\boldsymbol{\delta}$ according to the following proposition.

Proposition 4.4.1. *For $\boldsymbol{\delta} = \boldsymbol{\delta}_m$, the corresponding quasi-periodic function is given by*

$$F_{\boldsymbol{\delta}} := \mathcal{R}F_{\boldsymbol{\delta}} \circ \mathcal{R}\mathrm{Lie}^{-1}.$$

Thus, by definition, the following diagram commutes:



Proof. A. We first demonstrate that $\mathcal{R}F_{\boldsymbol{\delta}}$ satisfies the functional equation of $F_{\boldsymbol{\delta}}$, proceeding as in Remark 4.2.2. Using $m = \sum \delta_j(t) n_j$, we see that

$$\begin{aligned}\sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{t\alpha m}{t-\theta} \right) &= \sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{(t-\theta+\theta)\alpha m}{t-\theta} \right) \\ &= \sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} (\alpha m) + \sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{\theta \alpha m}{t-\theta} \right).\end{aligned}$$

Note by (4.6) and (4.7), that $m \in \tau M_f = \tau \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W)) \subset \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W - \Xi))$, and so $\mathrm{res}_{\Xi^{(0)}}(\alpha m) = 0$. Therefore, the above calculation continues

$$\begin{aligned}&= \sum_{i=0}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\alpha \sum_j \delta_j(t) n_j \right) + \sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{\theta \alpha m}{t-\theta} \right) \\ &= \boldsymbol{\delta}(t)(\mathcal{R}\mathrm{Exp}(\alpha)) + \theta \mathcal{R}F_{\boldsymbol{\delta}}(\alpha),\end{aligned}$$

where the last equality follows exactly as in (4.15), using the definitions of $\mathcal{R}\mathrm{Exp}$ and $\mathcal{R}F_{\boldsymbol{\delta}}$.

B. Next we show that the composite $\mathcal{R}F_{\boldsymbol{\delta}} \circ \mathcal{R}\mathrm{Lie}^{-1} : \mathrm{Lie}(E_f) \to C_\infty^d$ is an entire function. For that we choose a system $z_1, \ldots, z_d$ of coordinates for $\mathrm{Lie}(E_f)$. Given $\alpha \in \mathcal{R}$, set $z_j := \mathrm{res}_{\Xi^{(0)}}(\alpha n_j)$, $j = 1, \ldots, d$. Thus from the definitions of the various twists involved, we see that

$$z_j^{q^i} = \mathrm{res}_{\Xi^{(i)}}(\alpha^{(i)} n_j^{(i)}). \tag{4.22}$$

Our goal is to express $\mathrm{res}_{\Xi^{(i)}}(\alpha m/(t-\theta))$ as a C_∞-linear combination of $z_j^{q^i}$ with coefficients which decrease sufficiently rapidly with i.

We make some preliminary observations. Let $r := \alpha - \frac{\alpha^{(1)}}{g} \in \Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W+\Xi))$. Recall (4.16) and use the fact

$$\frac{r^{(j)}}{g^{(0)}\cdots g^{(j-1)}} \in \Gamma(\mathbf{U}, \mathcal{O}_{\mathbf{X}}(-W + \textstyle\sum_{l=0}^{j} \Xi^{(l)})),$$

which is verified recursively, to see that

$$\mathrm{res}_{\Xi^{(i)}}\left(\frac{\alpha m}{t-\theta}\right) = \mathrm{res}_{\Xi^{(i)}}\left(\frac{\alpha^{(i)}}{g^{(0)}\cdots g^{(i-1)}}\,\frac{m}{t-\theta}\right). \tag{4.23}$$

Next, we write m as

$$m = \sum_j \delta_j(t) n_j = \sum_{j=1}^{d}\sum_{s=1}^{u} \gamma_{sj}\tau^s n_j, \quad \gamma_{sj} \in C_\infty. \tag{4.24}$$

Combining (4.23) and (4.24) we then have

$$\mathrm{res}_{\Xi^{(i)}}\left(\frac{\alpha m}{t-\theta}\right) = \sum_{k=1}^{d}\sum_{s=1}^{u} \mathrm{res}_{\Xi^{(i)}}\left(\frac{\alpha^{(i)}}{g^{(0)}\cdots g^{(i-1)}}\,\frac{\gamma_{sk}}{t-\theta}\,\tau^s n_k\right). \tag{4.25}$$

Finally, since for each k, $1 \le k \le d$, we have $(t-\theta)n_k \in \tau M_f$, if we set $\mathbf{n} = (n_1, \dots, n_d)^{tr}$, we can write

$$(t-\theta)\mathbf{n} = B_1\tau\mathbf{n} + \cdots + B_v\tau^v\mathbf{n},$$

where each $B_k \in \mathrm{Mat}_{d\times d}(C_\infty)$. Then we divide by $t-\theta$ and substitute this expression in for $\mathbf{n}$ in the term $B_1\tau\mathbf{n}$ to obtain

$$\mathbf{n} = B_2^{[1]}\tau^2\mathbf{n} + \cdots + B_{v+1}^{[1]}\tau^{v+1}\mathbf{n},$$

where each $B_i^{[1]}\tau^i$ has entries which are the sum of terms of the form $b_i/(t-\theta)\tau^i$ or $b_1(t-\theta)^{-1}\tau\cdot((t-\theta)^{-1}b_{i-1})\tau^{i-1}$, where the b_i are coefficients of B_i. Continuing in this way, we can progressively eliminate the first $j-1$ degree terms to find an expression

$$\mathbf{n} = B_{j+1}^{[j]}\tau^{j+1}\mathbf{n} + \cdots + B_{v+j}^{[j]}\tau^{v+j}\mathbf{n}, \tag{4.26}$$

where each $B_w^{[j]}\tau^w$ has coefficients which are sums of terms of the form

$$\left(\frac{b_{e_1}}{t-\theta}\right)\tau^{e_1}\left(\frac{b_{e_2}}{t-\theta}\right)\tau^{e_2}\left(\frac{b_{e_3}}{t-\theta}\right)\tau^{e_3}\cdots\left(\frac{b_{e_l}}{t-\theta}\right)\tau^{e_l} \tag{4.27}$$

for which $e_1 + e_2 + \cdots + e_{l-1} + e_l = w$, for $1 \le e_i \le v$, each b_e is an entry of the constant matrix B_e, and $j+1 \le w \le j+v$.

Now we substitute the expressions from (4.26) with $j = i-s$ into the terms of (4.25) involving $\tau^s n_k$. According to (4.7), when we multiply n_k by the preceding expression (4.27) and apply τ^s, we obtain

$$\left(\frac{b_{e_1}}{t-\theta}\right)^{(s)}\left(\frac{b_{e_2}}{t-\theta}\right)^{(s+e_1)}\left(\frac{b_{e_3}}{t-\theta}\right)^{(s+e_1+e_2)}\cdots\left(\frac{b_{e_l}}{t-\theta}\right)^{(s+e_1+\cdots+e_{l-1})}\times \\ g^{(0)}\cdots g^{(w+s-1)} n_k^{(w+s)} \tag{4.28}$$

When $w > i - s$, the above differential multiplied by

$$\frac{\alpha^{(i)}}{g^{(0)} \dots g^{(i-1)}} \frac{1}{t-\theta}$$

is regular at $\Xi^{(i)}$, since then the uncancelled $g^{(i)}$ occurs in the numerator. Therefore the contribution towards the reside at $\Xi^{(i)}$ of such terms is nil, and we can concentrate on the terms with $w = i - s$. We obtain that

$$\mathrm{res}_{\Xi^{(i)}} \left(\frac{\alpha m}{t-\theta} \right) = \sum_{j=1}^{d} \mathrm{res}_{\Xi^{(i)}} (G_{j,i} \alpha^{(i)} n_j^{(i)}),$$

where $G_{j,i}$ is a rational function on $\mathbf{X}$ without poles along $\Xi^{(i)}$, and so by (4.22),

$$\mathrm{res}_{\Xi^{(i)}} \left(\frac{\alpha m}{t-\theta} \right) = \sum_{j=1}^{d} \sum_{x \in \Xi^{(i)}} G_{j,i}(x) z_j^{q^i}. \tag{4.29}$$

It follows that

$$\mathcal{R}F_{\boldsymbol{\delta}} \circ \mathcal{R}\mathrm{Lie}^{-1}(z_1, \dots, z_d) = \sum_{i=1}^{\infty} \sum_{j=1}^{d} \sum_{x \in \Xi^{(i)}} G_{j,i}(x) z_j^{q^i}, \tag{4.30}$$

and so, using the fact that we have already shown that this function satisfies the proper functional equation, by Proposition 3.2.1 it will be shown equal to $F_{\boldsymbol{\delta}}$ once we verify that the right-hand side is entire.

To this end we need to estimate $|G_{j,i}(x)|$, which we do by estimating the terms appearing in (4.28). Since we have the bound

$$\frac{\left| b_{e_k}^{(s+e_1+\dots+e_{k-1})} \right|}{\left| \theta^{q^i} - \theta^{(s+e_1+\dots+e_{k-1})} \right|} \leq \frac{B^{q^{s+\dots+e_{k-1}}}}{|\theta|^{q^i}}$$

on the factors occurring in (4.28), where B is an upper bound for the absolute values of the entries of the matrices B_j, we see that

$$|G_{j,i}(x)| \leq \frac{C}{|\theta|^{q^i}} \left| \frac{B^{q+q^2+\dots+q^{i-1}}}{(\theta^{q^i})^{\frac{i-s}{v}}} \right| \leq \frac{C}{|\theta|^{q^i}} \frac{B^{q^i}}{|\theta|^{\frac{i-s}{v} q^i}} \leq \frac{C_0^{q^i}}{|\theta|^{\frac{i}{v} q^i}},$$

where C is an upper bound on the coefficients γ_{sj} in (4.24). Here we have used the fact that $s + e_1 + \dots + e_l = i$ in (4.27), whereas $1 \leq e_j \leq v$, $1 \leq j \leq l$. From this estimate it is clear that (4.30) is entire, since $|\theta| > 1$. □

4.4.2. *Quasi-periods.* As defined in Section 3.2, the quasi-periods of E_f coming from $\boldsymbol{\delta}$ are the values $F_{\boldsymbol{\delta}}(\Lambda_f) \subset C_\infty$, where Λ_f is the period lattice of E_f. These quasi-periods (with signs reversed) are then the coordinates of periods of the quasi-periodic extension of E_f corresponding to $\boldsymbol{\delta}$.

The quasi-periods of any inner biderivation $\boldsymbol{\delta}$ are linear combinations of the coordinates of the periods of E_f with coefficients from any field of definition of E_f and $\boldsymbol{\delta}$. Thus for questions of linear independence over $\bar{k}$ involving both periods and quasi-periods, we may as well assume that our biderivations are strictly reduced and defined over $\bar{k}$. Our first task is to find a convenient C_∞-basis for $H_{sr}(\Phi_f)$ or, equivalently by (4.21), a basis for $\tau M_f/(t-\theta)M_f$ defined over $\bar{k}$.

Lemma 4.4.2. *There exists a C_∞-basis $\{n_a : a \in \mathcal{I} \smallsetminus \mathcal{I}_+\}$ for $\tau M_f/(t-\theta)M_f$ for which*

(a) *n_a is defined over $\overline{k}$;*

(b) *$n_a \in \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W - 2[\mathcal{I}]\xi + 2[a]\xi))$;*

(c) $\mathrm{res}_{[a]\xi}(n_a/(t-\theta)) = 1$.

Proof. The C_∞-linear maps $\left\{m \mapsto \mathrm{res}_{[a]\xi}\left(\frac{m}{t-\theta}\right) : a \in \mathcal{I} \smallsetminus \mathcal{I}_+\right\}$ on

$$\tau M_f = \tau\Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W)) = \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W - \Xi)),$$

are linearly independent over C_∞. Indeed by Riemann-Roch, for j sufficiently large $\Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(jI - 2[\mathcal{I}]\xi + [a]\xi)) \subsetneq \Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(jI - 2[\mathcal{I}]\xi + 2[a]\xi))$ and $\Gamma(\mathbf{X}, \Omega_{\mathbf{X}}(jI + W - \Xi - [a]\xi)) \subsetneq \Gamma(\mathbf{X}, \Omega_{\mathbf{X}}(jI + W - \Xi))$. Further they are trivial on $(t-\theta)M_f$. Thus we can choose $\{n_a\}$ as a dual basis modulo $(t-\theta)M_f$. That $\{n_a\}$ are defined over $\overline{k}$ follows from the fact that the t-motive M_f is obtained by extending scalars on a t-motive which is initially defined over $\overline{k}$ (as in Theorem 4.2.1). □

We now fix a basis $\{n_a : a \in \mathcal{I} \smallsetminus \mathcal{I}_+\}$ for $\tau M_f/(t-\theta)M_f$ as in the above lemma, thus obtaining a basis $\{\boldsymbol{\delta}_a\}$ for $H_{sr}(\Phi_f)$. For $b \in \mathsf{B}_f$, let

$$\boldsymbol{\lambda}_b := \begin{pmatrix} \vdots \\ \sigma_a(\iota(b))\pi_a \\ \vdots \end{pmatrix}_{a \in \mathcal{I}_+}$$

be a period in $\mathrm{Lie}(E_f)$ as in (4.18). We define for each $a \in \mathcal{I} \smallsetminus \mathcal{I}_+$,

$$\eta_{a,b} := \eta_a(\boldsymbol{\lambda}_b) := F_{\boldsymbol{\delta}_a}(\boldsymbol{\lambda}_b)$$

to be the corresponding quasi-periods.

Theorem 4.4.3. *Fix a basis for $\tau M_f/(t-\theta)M_f$ defined over $\overline{k}$ as in Lemma 4.4.2. For each $a \in \mathcal{I} \smallsetminus \mathcal{I}_+$ and $b \in \mathsf{B}_f$,*

$$\eta_{a,b} = F_{\boldsymbol{\delta}_a}(\boldsymbol{\lambda}_b) = \sigma_a(b) c_a \Gamma\left(\frac{a}{f}\right), \quad c_a \in \overline{k}^{\times}.$$

Remark. For an arbitrary Φ_f-biderivation defined over $\overline{k}$, the corresponding quasi-periods are $\overline{k}$-linear combinations of Gamma values in

$$\Gamma_f := \{\Gamma(a/f) : \deg(a) < \deg(f), (a,f) = 1\}.$$

Indeed, the strictly quasi-periodic extensions Q_f of E_f with $j = d - r(E_f)$ as described in Proposition 3.4.1 provide examples. Furthermore, Theorem 4.4.3 shows that by appropriately choosing a basis for $H_{sr}(\Phi_f)$, we guarantee that the periods of this quasi-periodic extension have coordinates which are simply non-zero algebraic multiples of all the values in Γ_f.

Remark. It is possible to define more general soliton functions on $\mathbf{X}$. Namely, given $s \in \mathsf{A}$ with $\deg(s) < \deg(f)$, Sinha defines a function ϕ_s on $\mathbf{X}$ which is a certain pull-back of ϕ (see [16, §2.2.8]). We can proceed with our various concerns using this function ϕ_s instead of ϕ and construct a corresponding t-module $E_{f,s}$. Nevertheless, this t-module can be shown to be isogenous to E_f, so for considerations of transcendence and linear independence over $\overline{k}$ for Gamma values, we gain nothing new.

Proof of Theorem 4.4.3. Fix $a \in \mathcal{I} \setminus \mathcal{I}_+$ and $b \in \mathsf{B}_f$. In the following, for precision we distinguish between the function $b \in \mathsf{B}_f$ on $\mathbf{X}$ and the scalar $\iota(b) \in B_f$. As $\boldsymbol{\lambda}_b = \mathcal{R}\mathrm{Lie}(b \cdot c_g)$ from (4.12), it follows from Proposition 4.4.1 that

$$\eta_{a,b} = \mathcal{R}F_{\boldsymbol{\delta}_a}(b \cdot c_g) = \sum_{i=1}^{\infty} \mathrm{res}_{\Xi^{(i)}} \left(\frac{bc_g n_a}{t-\theta} \right). \tag{4.31}$$

From Lemma 4.4.2 it follows that $bn_a/(t-\theta) \in \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W - [\mathcal{I}]\xi + 2[a]\xi))$. Thus as $c_g \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}(-W + \Xi))$, the remark immediately following equation (4.11) shows that the poles of $bc_g n_a/(t-\theta)$ contained in $\mathbf{U}$ lie along the support of the divisor

$$[a]\xi + \Xi^{(0)} + \Xi^{(1)} + \cdots .$$

Moreover, we establish

$$\mathrm{res}_{[a]\xi} \left(\frac{bc_g n_a}{t-\theta} \right) + \mathrm{res}_{\Xi^{(0)} + \Xi^{(1)} + \cdots} \left(\frac{bc_g n_a}{t-\theta} \right) = 0. \tag{4.32}$$

This equality follows from Lemma 4.2.4. The required estimates are obtained exactly as in Sinha [16, Lemma 4.6.4] (taking $\alpha = c_g$), and the sum on the left of (4.32) is taken over all of the poles of $bc_g n_a/(t-\theta)$ contained in $\mathbf{U}$. Combining (4.31) and (4.32) we find that

$$\eta_{a,b} = \mathcal{R}F_{\boldsymbol{\delta}_a}(b \cdot c_g) = -\,\mathrm{res}_{[a]\xi + \Xi^{(0)}} \left(\frac{bc_g n_a}{t-\theta} \right). \tag{4.33}$$

As $bn_a/(t-\theta) \in \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}(W - [\mathcal{I}]\xi + 2[a]\xi))$, it follows that $bc_g n_a/(t-\theta)$ is regular at $[a']\xi$ for $a' \in \mathcal{I}$, $a' \neq a$. Lemma 4.4.2bc combined with the calculation of (4.19) shows that

$$\mathrm{res}_{[a]\xi} \left(\frac{bc_g n_a}{t-\theta} \right) = \sigma_a(\iota(b)) g([a]\xi)^{-1} \frac{a}{f} \Gamma\left(\frac{a}{f} \right).$$

Because g and $[a]\xi$ are defined over $\overline{k}$, the constant $c_a := -g([a]\xi)^{-1}\frac{a}{f} \in \overline{k}$ and $c_a \neq 0$. □

4.5. **Sub-t-modules and Connections with Bracket Relations.** In this section we investigate the correspondence between the bracket relations on special values of the Gamma function of Section 1.2 and the presence of sub-t-modules in soliton t-modules E_f. Since the soliton t-modules are of CM-type, the results of Section 2.5 apply.

Let f be monic and let $a \in \mathcal{I}$. We define the following subset of $\mathrm{Gal}(\mathsf{K}_f/\mathsf{k})$:

$$\mathsf{F}(a) := \{\sigma_s \in \mathrm{Gal}(\mathsf{K}_f/\mathsf{K}) : \Gamma(s/f) \approx \Gamma(a/f)\}. \tag{4.34}$$

Note first that $\mathsf{F}(1)$ is in fact a subgroup of $\mathrm{Gal}(\mathsf{K}_f/\mathsf{k})$: by the bracket relations, with $m_1 = 1$, $m_s = -1$ and all other entries of $\mathbf{m}$ equal to 0, we see $\sigma_s \in \mathsf{F}(1)$ if and only if for all representatives u of elements of $(\mathsf{A}/f)^\times$ we have

$$us \bmod f \text{ is monic} \iff u \bmod f \text{ is monic},$$

where $a \bmod f$ denotes the remainder of a after division by f. This condition is certainly closed under multiplication. Similarly, for each $a \in \mathcal{I}$, we find that $\mathsf{F}(a)$ is a coset of $\mathsf{F}(1)$, i.e. $\mathsf{F}(a) = \mathsf{F}(1)\sigma_a$.

The set $\mathcal{S}_f \subset \mathrm{Gal}(\mathsf{K}_f/\mathsf{k})$ from Remark 4.3.2 is defined by

$$\mathcal{S}_f = \{\sigma_a \in \mathrm{Gal}(\mathsf{K}_f/\mathsf{k}) : a \bmod f \text{ is monic}\}, \tag{4.35}$$

and so we find that $\mathcal{S}_f$ is the union of cosets of $\mathsf{F}(1)$. Let $\mathsf{L}_f \subset \mathsf{K}_f$ be the fixed field of $\mathsf{F}(1)$. Theorem 2.5.2 then shows that there is a sub-t-module H_f of CM-type $(\mathsf{L}_f, \mathcal{S}_f|_{\mathsf{L}_f})$ such that E_f is isogenous to H_f^m, where $m = [\mathsf{K}_f : \mathsf{L}_f]$.

Proposition 4.5.1. *The relation on $\mathcal{I}$ induced by $\Gamma(a/f) \approx \Gamma(b/f)$ gives a decomposition of $\mathcal{I}$ into disjoint subsets of cardinality m.*

Proof. $|\mathsf{F}(a)| = |\mathsf{F}(1)| = [\mathsf{K}_f : \mathsf{L}_f] = m$. □

Lemma 4.5.2. *H_f is a simple t-module.*

Proof. Since H_f is itself a t-module of CM-type, it is isogenous to a power of a simple t-module of CM-type. Thus there is a smallest field $\mathsf{L} \subset \mathsf{L}_f \subset \mathsf{K}_f$ which satisfies the criteria of Lemma 2.5.1. Let F be the subgroup of $\mathrm{Gal}(\mathsf{K}_f/\mathsf{k})$ corresponding to $\mathrm{Gal}(\mathsf{K}_f/\mathsf{L})$. Certainly $\mathsf{F}(1) \subset \mathsf{F}$. However, if $\sigma_s \in \mathsf{F}$, then since $\mathcal{S}_f$ is the union of cosets of F, it must be the case for all $u \in (\mathsf{A}/f)^\times$ that $us \bmod f$ is monic if and only if $u \bmod f$ is monic. Thus $\mathsf{F} = \mathsf{F}(1)$, $\mathsf{L} = \mathsf{L}_f$, and H_f is simple. □

The following proposition follows from Proposition 4.5.1 and the simplicity of H_f.

Proposition 4.5.3. *The soliton t-module E_f has a proper sub-t-module if and only if there exist distinct a, $b \in \mathcal{I}$ such that $\Gamma(a/f) \approx \Gamma(b/f)$.*

We now consider special values of the Gamma function at fractions having different denominators. When necessary we write for any $n \in \mathsf{A}_+$, $\mathcal{I}_n = \{a \in \mathsf{A} : \deg(a) < \deg(n), (a,n) = 1\}$, and similarly for $\mathsf{F}_n(a)$ corresponding to the group in (4.34).

Theorem 4.5.4. *Let f and g be monic and distinct. The following are equivalent.*

(a) *There exist non-trivial t-module homomorphisms $E_f \to E_g$.*
(b) *H_f and H_g are isogenous.*
(c) *There exist $a \in \mathcal{I}_f$ and $b \in \mathcal{I}_g$ such that $\Gamma(a/f) \approx \Gamma(b/g)$.*

Proof. Certainly (a) and (b) are equivalent according to the discussion in Section 2.5. Assuming that H_f and H_g are isogenous, then by Theorem 2.5.3 the CM-field L of H_f and H_g is a subfield of $\mathsf{K}_f \cap \mathsf{K}_g$ and simultaneously satisfies the criteria of Lemma 2.5.1 for both K_f and K_g. Furthermore, if we let $\mathcal{S}_\mathsf{L}$ be the preferred embeddings for the CM-type of H_f, then for some $\sigma_b \in \mathrm{Gal}(\mathsf{K}_g/\mathsf{k})$ we have

$$\mathcal{S}_\mathsf{L} = \mathcal{S}_f|_\mathsf{L} = (\mathcal{S}_g|_\mathsf{L})\sigma_b^{-1}|_\mathsf{L}. \tag{4.36}$$

Now let m be the least common multiple of f and g, and let $u \in \mathcal{I}_m$. We claim that

$$u \bmod f \text{ is monic} \iff ub \bmod g \text{ is monic}. \tag{4.37}$$

Indeed, suppose $u \bmod f$ and $\zeta ub \bmod g$ are monic with $\zeta \in \mathbb{F}_q^\times$. It follows that $\sigma_u \in \mathcal{S}_f$ and $\sigma_{\zeta ub} \in \mathcal{S}_g$. However, by (4.36) we can choose $\sigma_y \in \mathcal{S}_g$ so that $\sigma_y\sigma_b^{-1}|_\mathsf{L} = \sigma_u|_\mathsf{L}$. Then certainly $\sigma_y\sigma_b^{-1}|_{\mathsf{L}_+} = \sigma_{\zeta ub}\sigma_b^{-1}|_{\mathsf{L}_+}$, and so by the hypotheses on L from Lemma 2.5.1, we must have

$$\sigma_y|_\mathsf{L} = \sigma_{\zeta ub}|_\mathsf{L}.$$

Therefore $\sigma_{\zeta u}|_\mathsf{L} = \sigma_y\sigma_b^{-1}|_\mathsf{L} = \sigma_u|_\mathsf{L}$, implying that $\zeta = 1$. Using (4.37) we find that $\Gamma(1/f) \approx \Gamma(b/g)$, completing (b) implies (c).

Now suppose that $\Gamma(a/f) \approx \Gamma(b/g)$ for some $a \in \mathcal{I}_f$ and $b \in \mathcal{I}_g$. For any $u \in \mathcal{I}_m$ it follows that $\Gamma(au/f) \approx \Gamma(bu/g)$. Thus we can specify without loss of generality that $\Gamma(1/f) \approx \Gamma(b/g)$ for some $b \in \mathcal{I}_g$. Let $\mathsf{G} \subset \mathrm{Gal}(\mathsf{K}_m/\mathsf{k})$ be the subset

$$\mathsf{G} := \{\sigma_s \in \mathrm{Gal}(\mathsf{K}_m/\mathsf{k}) : \Gamma(s/f) \approx \Gamma(1/f)\}.$$

Certainly G is a subgroup of $\mathrm{Gal}(\mathsf{K}_m/\mathsf{k})$ by the bracket relations. We claim that

$$\mathsf{G} = \{\sigma_s \in \mathrm{Gal}(\mathsf{K}_m/\mathsf{k}) : \Gamma(bs/g) \approx \Gamma(b/g)\}.$$

Indeed, if $u \in (\mathsf{A}/m)^\times$ and $\sigma_s \in \mathsf{G}$, then

$$\begin{array}{ccc} ub \bmod g \text{ monic} & \overset{(4.37)}{\Longleftrightarrow} & u \bmod f \text{ monic} \\ {\scriptstyle \sigma_s|_{K_g} \in \mathsf{F}_g(1)} \Big\Updownarrow & & \Big\Updownarrow {\scriptstyle \sigma_s|_{K_f} \in \mathsf{F}_f(1)} \\ ubs \bmod g \text{ monic} & \underset{(4.37)}{\Longleftrightarrow} & us \bmod f \text{ monic}. \end{array}$$

Furthermore, as in (4.34),

$$\mathsf{G}|_{\mathsf{K}_f} = \mathsf{F}_f(1) \quad \text{and} \quad \mathsf{G}|_{\mathsf{K}_g} = \mathsf{F}_g(1).$$

We let L be the fixed field of G, and thus L is the CM-field of both H_f and H_g. Because $(\mathsf{A}/m)^\times \to (\mathsf{A}/f)^\times$ and $(\mathsf{A}/m)^\times \to (\mathsf{A}/g)^\times$ are surjective, if follows from (4.35) that

$$\mathcal{S}_f = \{\sigma_u|_{\mathsf{K}_f} : u \in (\mathsf{A}/m)^\times \text{ and } u \bmod f \text{ is monic}\},$$
$$\mathcal{S}_g = \{\sigma_u|_{\mathsf{K}_g} : u \in (\mathsf{A}/m)^\times \text{ and } u \bmod g \text{ is monic}\}.$$

Because $\Gamma(1/f) \approx \Gamma(b/g)$, it follows that

$$\sigma_u|_{\mathsf{K}_f} \in \mathcal{S}_f \Longleftrightarrow \sigma_u|_{\mathsf{K}_g}\sigma_b \in \mathcal{S}_g.$$

Thus $\mathcal{S}_f|_{\mathsf{L}} = \mathcal{S}_g|_{\mathsf{L}}\sigma_b^{-1}|_{\mathsf{L}}$, and by Theorem 2.5.3 H_f is isogenous to H_g. □

Remark. The proofs of the above results make no use of the period computations for E_f and Q_f performed in the previous sections. With these period computations in hand, it should be pointed out that Yu's Theorem of the Sub-t-module provides another explanation for the direction that the existence of bracket relations of the form $\Gamma(a/f) \approx \Gamma(b/g)$ guarantees that H_f and H_g are isogenous. These methods will be the main focus of the next section.

5. Linear Independence Results

5.1. **Proofs of Results.** In this section, we apply the results of preceding sections to obtain linear independence statements for minimal quasi-periodic extensions of simple t-modules, for t-modules of CM-type in general, and for soliton t-modules in particular. As the setting becomes more and more specialized, we shall see that the assertions become more specific. In this section, for a t-module E defined over $\overline{k}$, we denote by Q_E a strictly quasi-periodic extension of maximal dimension and defined over $\overline{k}$.

Theorem 5.1.1. *Let H be a simple t-module defined over $\overline{k}$ in which $d\Phi_H(t) = \theta I_d$. Let $Q := Q_H = (\Psi, \mathbb{G}_a^{d+j})$ be a strictly quasi-periodic extension of H with* *corresponding quasi-periodic functions* $F_1, \ldots, F_j$. *Let* $\mathbf{u} = (u_1, \ldots, u_d) \in C_\infty^d$ *be non-zero with* $\operatorname{Exp}_H(\mathbf{u}) \in \overline{k}^d$. *Then the quantities*

$$u_1, \ldots, u_d, F_1(\mathbf{u}), \ldots, F_j(\mathbf{u}) \tag{5.1}$$

are $\overline{k}$*-linearly independent.*

Proof. If the values of (5.1) are linearly dependent over $\overline{k}$, then, by Yu's Theorem of the Sub-t-Module [25, Theorem 3.3], the point corresponding to (5.1) lies in the tangent space at the origin of a sub-t-module R of Q.

Proposition 3.4.1 shows that Q is a minimal extension of H. Therefore, by minimality, $\mathbf{u}$ is contained in the tangent space of a proper sub-t-module S of H. As H is simple, S is zero, contrary to our choice of a non-zero $\mathbf{u}$. □

The proof of this result generalizes to admit several H and various $\mathbf{u}$. We say that the points $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \operatorname{Lie}(E)$ are *linearly independent over* $\operatorname{End}(E)$ when the only endomorphisms $e_1, \ldots, e_n \in \operatorname{End}(E)$ with $de_1\mathbf{u}_1 + \cdots + de_n\mathbf{u}_n = 0$ are $e_1 = \cdots = e_n = 0$.

Corollary 5.1.2. *Let* $H_1, \ldots, H_n$ *denote non-isogenous simple* t*-modules defined over* $\overline{k}$ *and with each* $d\Phi_i(t) = \theta I_{d_i}$. *For each* $i = 1, \ldots, n$, *let the set* $\mathcal{U}_i = \{\mathbf{u}_{i1}, \ldots, \mathbf{u}_{i\ell}\}$ *be linearly independent over* $\operatorname{End}(H_i)$, $\mathbf{u}_{i\ell_i} = (u_{i\ell 1}, \ldots, u_{i\ell d_i})$ *with each* $\operatorname{Exp}_{H_i}(\mathbf{u}_{il}) \in \mathbb{G}_a(\overline{k})^{d_i}$, *and let* $F_{i1}, \ldots, F_{ir_i}$ *defined over* $\overline{k}$ *be the quasi-periodic functions corresponding to a maximal strictly quasi-periodic extension* Q_{H_i} *over* $\overline{k}$. *Then the quantities*

$$u_{i\ell m}, \;\; F_{jh}(\mathbf{u}_{\ell'})$$

are $\overline{k}$*-linearly independent.*

Proof. By Yu's Theorem of the Sub-t-Module [25, Theorem 3.3], any linear dependence relation would give a proper sub-t-module R of $\prod Q_{H_i}^{\ell_i}$ with $\operatorname{Lie}(R)$ containing the point whose coordinates are given by tuples of the above values. Since strictly quasi-periodic extensions are minimal, we know by Proposition 1 of [6] that R projects onto a proper sub-t-module S of $\prod H_i^{\ell_i}$.

In that case, we can apply Yu's Kolchin-type result [25, Theorem 1.3] to conclude that, for some fixed i, there are non-zero endomorphisms $\Theta_\ell \in \operatorname{End}(H_i), \ell = 1, \ldots, \ell_i$ such that the projection of S is contained in the sub-t-module of $H_i^{\ell_i}$ given by $\sum_\ell \Theta_\ell \mathbf{x}_\ell = 0$.

In particular, $\sum d\Theta_\ell \mathbf{u}_{i\ell} = 0$. This identity is contradicted by our hypothesis that $\mathbf{u}_{i1}, \ldots, \mathbf{u}_{i\ell_i}$ are linearly independent over $\operatorname{End}(H_i)$. □

Recall that, according to Theorem 2.5.2, every t-module E of CM-type is isogenous to a power of a simple sub-t-module H (also of CM-type). So the above result tells us in particular that the coordinates of periods of Q_E are given by those of Q_H. However in the CM setting, we can be even more specific.

Theorem 5.1.3. *Let* E *be a* t*-module of CM-type* $(\mathsf{K}, \mathcal{S})$ *defined over* $\overline{k}$. *Say that* E *is isogenous to* H^m *with* H *simple, defined over* $\overline{k}$, *and of CM-type* $(\mathsf{L}, \mathcal{S}|_{\mathsf{L}})$. *Let* Q_E *and* Q_H *denote maximal strictly quasi-periodic extensions of* E *and* H, *respectively, defined over* $\overline{k}$. *Then the* $\overline{k}$*-vector space* V_E *spanned by the coordinates of all periods of* Q_E *has as basis the coordinates of* any *non-zero period of* Q_H.

Proof. Note from Corollaries 3.5.3, 3.5.4 that V_E is spanned by the coordinates of the periods from Q_H. We use the CM structure to show that V_{Q_H} is spanned by the coordinates of any non-zero period of Q_H:

Define $\boldsymbol{\sigma}_H := \boldsymbol{\sigma}_{\mathcal{S}}|_L$, in the notation of Section 2.3. By Remark 2.3.1, we may assume that the period lattice of H is $\boldsymbol{\sigma}_H(\mathsf{B}_H)\boldsymbol{\lambda}$ for some $\boldsymbol{\lambda}$. We need to verify that, for each $\boldsymbol{\delta}$ among the Φ_H-biderivations $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_j$ underlying the quasi-periodic extension Q_H, $F_{\boldsymbol{\delta}}(\boldsymbol{\sigma}_H(\mathsf{B}_H)\boldsymbol{\lambda})$ lies in the $\overline{k}$-span of the coordinates of $\boldsymbol{\lambda}$ and the $F_{\boldsymbol{\delta}_i}(\boldsymbol{\lambda})$.

Recall that Φ_H extends from A to B_H in such a way that

$$\Phi_H(t)\Phi_H(b) = \Phi_H(b)\Phi_H(t)$$

and thus in particular $\boldsymbol{\sigma}_H(b)d\Phi_H(t) = d\Phi_H(t)\boldsymbol{\sigma}_H(b)$. Now let $\boldsymbol{\delta}$ be a Φ_H-biderivation. Then the fact that

$$\begin{aligned} F_{\boldsymbol{\delta}}(\boldsymbol{\sigma}_H(b)d\Phi_H(t)\mathbf{z}) &= F_{\boldsymbol{\delta}}(d\Phi_H(t)\boldsymbol{\sigma}_H(b)\mathbf{z}) \\ &= \theta F_{\boldsymbol{\delta}}(\boldsymbol{\sigma}_H(b)\mathbf{z}) + \boldsymbol{\delta}(t)\,\mathrm{Exp}_H(\boldsymbol{\sigma}_H(b)\mathbf{z}) \\ &= \theta F_{\boldsymbol{\delta}}(\boldsymbol{\sigma}_H(b)\mathbf{z}) + \boldsymbol{\delta}(t)\Phi_H(b)\,\mathrm{Exp}_H(\mathbf{z}) \end{aligned}$$

shows thus that $F_{\boldsymbol{\delta}}(\boldsymbol{\sigma}_H(b)\mathbf{z})$ is itself the quasi-periodic function associated to the Φ_H-biderivation $\Phi(b)_*\boldsymbol{\delta}$. Therefore the values of this function at $\boldsymbol{\lambda} \in \Lambda$ will lie in the $\overline{k}$-span of the coordinates of any non-zero period. Hence V_{Q_H} is spanned by the coordinates of any non-zero period of Q_H.

Since the t-action on $\mathrm{Lie}(Q_H)$ is diagonal, J. Yu's Theorem of the Sub-t-Module [25, Theorem 3.3] implies that any $\overline{k}$-linear relation on the coordinates of a fixed period $\boldsymbol{\lambda}$ actually will hold on $\mathrm{Lie}(R)$ for some proper sub-t-module R of Q_H. However Q_H is a minimal extension of the t-module H. So the period onto which $\boldsymbol{\lambda}$ projects in H would lie in a proper sub-t-module of H. Since H is simple, $\boldsymbol{\lambda}$ projects onto 0; however such a $\boldsymbol{\lambda} = 0$, contrary to our hypothesis. Therefore the coordinates of $\boldsymbol{\lambda}$ and the quasi-periods $F_{\boldsymbol{\delta}_i}(\boldsymbol{\lambda})$ form an L-basis for V_H, as claimed. □

The preceding theorem also extends to several t-modules at once.

Theorem 5.1.4. *Let $E_1, \dots, E_n$ be t-modules of CM-type defined over $\overline{k}$. For each i, we use the following notation:*

(a) *$E_i \sim H_i^{m_i}$, with H_i simple, defined over $\overline{k}$.*

(b) *P_i denotes the set of coordinates of a non-zero period of Q_{H_i}.*

(c) *V_i denotes the $\overline{k}$-vector space spanned by all the non-zero coordinates of periods of Q_{E_i}.*

If the H_i are pair-wise non-isogenous, then $\cup_{i=1}^n P_i$ is a $\overline{k}$-basis of $V_1 + \cdots + V_n$.

Proof. We know from the preceding result that P_i is a $\overline{k}$-basis for the $\overline{k}$-vector space $V_i, i = 1, \dots, n$. Since each Q_{H_i} is a minimal extension of H_i, then by Lemma 1 of [6], $\prod Q_{H_i}$ is a minimal extension of $\prod H_i$. We know by Yu's Theorem of the Sub-t-Module that any $\overline{k}$-linear relation on $\cup P_i$ gives rise to a proper sub-t-module of $\prod Q_{H_i}$, which, by minimality, projects onto a proper sub-t-module R of $\prod H_i$. By the simplicity and non-isogeneity of the H_i, we know that the only proper sub-t-modules for $\prod H_i$ have trivial projections onto some factor, say H_j. But then the underlying period of H_j must be zero. That forces the whole period of Q_{H_j} to vanish, contrary to our choice of P_j. □

The above considerations apply to soliton t-modules. But we can be completely precise in this case. For the following theorem we recall that the notation $\Gamma(r_1) \approx \Gamma(r_2)$ means that the bracket relations imply that $\Gamma(r_1)/\Gamma(r_2) \in \overline{k}$.

Theorem 5.1.5. *The numbers*

$$\{1, \widetilde{\pi}\} \cup \mathrm{rep}\{\Gamma(r) : r \in k \smallsetminus A\}/\approx$$

are $\overline{k}$-linearly independent, where the notation means that we take any set of representatives of the equivalence classes for the relation $\approx$ on the set $\{\Gamma(r) : r \in k \smallsetminus A\}$.

Proof. It is clear that the claim holds for every choice of representatives if it holds for any choice. If $r = b + a/f,\ a, b \in A, f \in A_+$, then $\Gamma(r) \approx \Gamma(a/f)$, since $a \bmod f = (bf + a) \bmod f$. Therefore we may always choose representatives of $\approx$ of the form $\Gamma(a/f),\ a \in \mathcal{I}_f, f \in A_+$.

Proposition 4.5.1 says that all equivalence classes among these values have cardinality m when $E_f \sim H_f^m$. Thus there are exactly $\dim Q_{H_f}$ classes among them. According to equation (4.20), Theorem 4.4.3, and Theorem 5.1.3, these representatives span V_{E_f}, a $\overline{k}$-space of dimension $\dim Q_{H_f}$. Therefore any choice $\mathrm{rep}(Q_{H_f})$ of $\approx$-representatives among the $\Gamma(a/f)$ gives a $\overline{k}$-basis for $V_{E_f} = V_{H_f}$.

Thus one choice of $\mathrm{rep}\{\Gamma(r) : r \in k \smallsetminus A\}/\approx$ will be a disjoint union of $\mathrm{rep}(Q_{H_f})$, taken over non-isogenous H_f. Consequently, the claim of the theorem is equivalent to the statement that $1, \widetilde{\pi}$ and coordinates of non-zero periods from non-isogenous Q_{H_f} are $\overline{k}$-linearly independent. We proceed to prove this assertion.

Yu's Theorem of the Sub-t-Module [25, Theorem 3.3] shows that, if there were a $\overline{k}$-linear relation on the coordinates of such periods, then there would have to be a proper sub-t-module R of a finite product of the form

$$Q := \mathbb{G}_a \times C \times Q_{H_{f_1}} \times \cdots \times Q_{H_{f_n}}$$

for which $\mathrm{Lie}(R)$ would contain the point $\mathbf{q} := (1, \widetilde{\pi}, \boldsymbol{\lambda}_{f_1}, \ldots, \boldsymbol{\lambda}_{f_n})$, where each $\boldsymbol{\lambda}_f$ is some non-zero period of Q_{H_f}, $f = f_1, \ldots, f_n$ and C denotes the Carlitz module. Here the Q_{H_f} occur exactly when some $\Gamma(a/f), a \in \mathcal{I}_f$ is involved in the supposed $\overline{k}$-linear relation. Moreover $\mathbb{G}_a$ and/or C appear only if 1 and/or $\widetilde{\pi}$ are involved in the relation. For ease of exposition, we simply assume that to be the case here.

By Lemma 1 of [6], the above product is minimal. Therefore R projects onto a proper sub-t-module S of the corresponding product

$$E := \mathbb{G}_a \times C \times H_{f_1} \times \cdots \times H_{f_n}$$

for which $\mathrm{Lie}(S)$ would contain the projection $\mathbf{p}$ of the point of $\mathbf{q}$ of $\mathrm{Lie}(R)$. Again the point $\mathbf{p}$ in $\mathrm{Lie}(S)$ projects non-trivially onto the factors because, as we saw in Proposition 3.3.1, the non-zero periods of Q_H are produced from – and project to – non-zero periods of H.

For the conclusion of the proof we keep in mind the following three remarks:

(a) The underlying simple H_f, H_g have been taken to be non-isogenous.

(b) The Carlitz module C has period an algebraic multiple of $\widetilde{\pi}$, and C does not have CM. Therefore it is a simple t-module which is not isogenous to any of the soliton t-modules E_f.

(c) The trivial t-module $\mathbb{G}_a$ is also simple and not isogenous to C nor any H_f.

Now

$$\mathrm{Lie}(E) = \mathrm{Lie}(\mathbb{G}_a) \times \mathrm{Lie}(C) \times \mathrm{Lie}(H_{f_1}) \times \cdots \times \mathrm{Lie}(H_{f_n}).$$

Since the factors of E are non-isogenous and simple, the only proper sub-t-modules have tangent spaces which project trivially to the tangent space of at least one factor. Thus $\mathrm{Lie}(S)$ does so, as S is a proper sub-t-module. However this is contradicted by the facts that $\mathbf{p} \in \mathrm{Lie}(S)$ and our choice of product Q ensures that $\mathbf{p} \in \mathrm{Lie}(S)$ has a non-zero entry in every factor of $\mathrm{Lie}(E)$. We conclude that there is no non-trivial $\overline{k}$-linear relation on the set in question. □

Remark. We note that, in the results of this section, we can adjoin coordinates of linearly independent logarithms of algebraic points, e.g. periods, of other simple t-modules defined over $\overline{k}$, as long as the t-modules are not isogenous to each other nor to the H_f nor the Carlitz module.

5.2. **Proof of Corollary 1.3.4.** As Corollary 1.3.2 is immediate from the Main Theorem, we need only consider Corollary 1.3.4. We break the proof up into several cases: we may without loss of generality consider f and the f_i to be monic.

5.2.1. *Case* $f = f_1$. Let $\mathsf{F} \subset (\mathsf{A}/f)^\times$ correspond to the Galois group $\mathrm{Gal}(\mathsf{K}_f/\mathsf{L})$ be a subgroup satisfying condition (b) of Lemma 2.5.1, where $\mathcal{S}_f \simeq \mathcal{I}_+$. For any $b \in \mathsf{F}$,

$$b\mathcal{I}_+ = \mathcal{I}_+,$$

as $\mathcal{I}_+$ is a union of cosets of F. Now

$$S := \sum_{a \in \mathcal{I}_+} a = 1,$$

as the sum of all monic polynomials over $\mathbb{F}_q$ of fixed positive degree with all but the constant term fixed is easily seen to vanish. Thus

$$b = bS = S = 1,$$

and F is trivial. We conclude from part (b) of Theorem 2.5.2 that E_f is simple. See Shimura [18, p. 64] for the analogue of this case for abelian varieties with CM by $\mathbb{Q}(\zeta_p)$. As E_f is simple and f is irreducible, Proposition 4.5.3 and Theorem 1.3.1 show that all the quantities $\Gamma(a/f)$, $\deg a < \deg f$ are $\overline{k}$-linearly independent.

5.2.2. *Case* $f = f_1^{e_1}$, $e_1 \geq 2$. The proof begins as before, except that now we must exclude the monic multiples of f_1 of degree less than $\deg f$ from the sum S. The argument given in the first case shows that the multipliers involved in these multiples add up to 1 and therefore the multiples themselves sum to f_1. Hence

$$S := \sum_{a \in \mathcal{I}_+} a = 1 - f_1.$$

As before, any $b \in \mathsf{F}$ satisfies

$$bS \equiv S \mod f,$$

and, since $(S, f) = 1$, we conclude $b = 1$. Thus as in the previous case, E_f is simple, and the $\Gamma(a/f)$, $\deg a < \deg f$, are $\overline{k}$-linearly independent.

However this does not cover the values $\Gamma(a/f)$ where $f_1 \mid a$. For them consider the t-modules $E_{f_1}, E_{f_1^2}, \ldots, E_{f_1^{e_1-1}}$, which are simple and according to their dimensions are non-isogenous. Thus from Theorems 4.5.4 and 1.3.1 we obtain the linear independence of all $\Gamma(a/f)$ for $0 \le \deg a < \deg f$, whether $(a, f) = 1$ or not.

5.2.3. *General case* $f = f_1^{e_1} \dots f_m^{e_m}$. Inclusion-exclusion gives that

$$S := \sum_{a \in I_+} a \equiv (1 - f_1) \cdots (1 - f_m) \mod f$$

(with equality unless $f = f_1 \cdots f_m$). Thus by hypothesis on the f_i, $(S, f) = 1$, whereas $bS = S$. So $b = 1$ and E_f is simple. This accounts for the linear independence of all $\Gamma(a/f)$, $a \in \mathcal{I}_f$.

The values $\Gamma(a/f)$ with $(a, f) \neq 1$ are included by induction through the remark that, for non-constant proper monic divisors $g \mid f$, the various E_g are simple and, since they have distinct CM-fields, are non-isogenous to each other and to E_f. The corollary follows.

6. Examples

Many of the calculations below on Gamma values are due to Sinha [15, §VI.3.2] and Thakur [19, §9]. The reader interested in the explicit computation of the Anderson-Coleman soliton functions should consult the specific examples of Coleman [7] and Sinha [16, §3.3] and the general methods of Thakur [21]. The examples below demonstrate in particular the correspondence between $\overline{k}$-linear relations on Gamma values and the structure of the underlying soliton t-modules developed in this paper.

6.1. **The Simplest Case.** Let $f = t$. In this case $\mathcal{I}_+ = \{1\}$, and so the t-module E_f is simply a Drinfeld module as it has dimension 1.

Here $\mathsf{K}_f = \mathsf{k}(z)$ and $\mathsf{B}_f = \mathsf{k}[z]$, where $z := \zeta_t = \sqrt[q-1]{-t}$, and thus the curve $\mathbf{X}$ is isomorphic to $\mathbb{P}^1/C_\infty$. Note that the point $\xi \in \mathbf{X}(C_\infty)$ corresponds to the zero of $z - \zeta_\theta$. Now the soliton function is

$$g_f = 1 - \frac{z}{\zeta_\theta},$$

and by (4.5) we have $W = 0$, $\Xi = \xi$ and $I = \infty$. The t-motive M_f is then

$$M_f = \Gamma(\mathbf{U}, \Omega_{\mathbf{X}}) = C_\infty[z]dz.$$

As defined in Section 4.3 we let $n_1 = \zeta_\theta^{q-2}\, dz$ and then note by (4.7) that $zn_1 = \left(\zeta_\theta \tau^0 + \frac{\zeta_\theta}{\theta^{q-2}} \tau\right) n_1$, and thus

$$\Phi(z) = \zeta_\theta \tau^0 + \frac{\zeta_\theta}{\theta^{q-2}} \tau. \tag{6.1}$$

Thus E_f is isomorphic over K_f to the Carlitz module for the polynomial ring $\mathbb{F}_q[z]$. By Theorem 4.3.1 we see that $\pi_1 = \Gamma(1/\theta)$. On the other hand, if $\widetilde{\pi}_\theta$ is the period of the Carlitz module for $\mathbb{F}_q[z]$, then we find from (6.1) that

$$\pi_1 = \Gamma\left(\frac{1}{\theta}\right) = \frac{\theta}{\sqrt[q-1]{\theta \zeta_\theta}} \widetilde{\pi}_\theta. \tag{6.2}$$

We now compute the quasi-periods for E_f. For each $\ell \in \mathbb{F}_q^\times$, $\ell \neq 1$, we choose n_ℓ as in Lemma 4.4.2. Namely we let

$$n_\ell := \frac{\ell}{\zeta_\theta^{q-2}} \left(\frac{t - \theta}{z - \ell \zeta_\theta}\right)^2 dz.$$

By Theorem 4.4.3 and in particular (4.33), we see that $\eta_\ell = -\operatorname{res}_{[\ell]\xi}(c_g n_\ell/(t-\theta))$. Proceeding as in (4.19), we obtain the quasi-periods

$$\eta_\ell = \frac{1}{\ell - 1} \cdot \frac{\ell}{\theta} \Gamma\left(\frac{\ell}{\theta}\right), \quad \ell \in \mathbb{F}_q^\times, \ell \neq 1.$$

Note that Corollary 1.3.3 shows that the numbers $\Gamma(\ell/\theta)$, $\ell \in \mathbb{F}_q^\times$, are $\overline{k}$-linearly independent.

6.2. **A Non-Uniform Example:** $f = t(t-1)$**.** We saw in the proof of Corollary 1.3.4 that, in order that E_f be non-simple when $f = f_1^{e_1} \dots f_m^{e_m}$, we must have some $f_i|(1-f_j)$. In this example, we will see that this necessary condition is not uniformly sufficient even in the simplest case, namely $f = t(t-1)$.

We look for possible subgroups F of $\mathcal{I}_+$. Since f is quadratic, we consider monic linear polynomials $b = t + \ell \in \mathsf{F}$, $\ell \in \mathbb{F}_q$. Then

$$b^2 \in \mathsf{F} \Leftrightarrow \begin{cases} 1 + 2\ell = 0 & \text{if } \ell^2 = 1, \\ 2\ell = 0 & \text{otherwise.} \end{cases}$$

In the first case, $4 = 1$, so $p = 3$. Since $(b, f) = 1$, $a \neq 0$, so in the second case $p = 2$. Thus, for $p > 3$, the t-module E_f is simple.

Let us examine the first case more closely when $p = 3$ under the assumption that E_f is non-simple. Then $\ell = 1$, so $\mathsf{F} = \{1, t+1\}$. Now as

$$\mathcal{I}_+ = \{1\} \cup \{t + a : a \in \mathbb{F}_q, a \neq 0, 1\} \tag{6.3}$$

is a union of cosets of F, we know that $(t+1)\mathcal{I}_+ = \mathcal{I}_+$. In particular for any element $t + a \neq t + 1$ of $\mathcal{I}_+$, $(t+a)(t+1) \in \mathcal{I}_+ \setminus \{1\}$. This means that $2 + a = 1$, and so $t + a = t - 1$. But $(f, t-1) \neq 1$. So in fact, if E_f is non-simple and $p = 3$, then $\mathcal{I}_+ = \mathsf{F} = \{1, t+1\}$, and obviously from (6.3), $q = 3$. Thus by Theorem 2.5.2, E_f is isogenous to a power of a Drinfeld module. The lattice for this Drinfeld module can be taken to be the ring of integers in the fixed field $\mathsf{k}(\zeta_t)$ of F, which in this case is $\mathbb{F}_3[\zeta_t] = \mathbb{F}_3[\sqrt{-t}]$.

When $p = 2$, the t-module E_f is 1-dimensional if $q = 2$. For $q > 2$, if $t + a$, $t + b$ are elements of F, then the closure of f under multiplication requires that the product $(t+a)(t+b)$ be monic modulo f of degree either one or zero. In the first case, $a = b$; in the second

$$1 + a + b = 0, \quad ab = 1,$$

i.e. $a^2 + a + 1 = 0$, i.e. $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$. Thus $q = 4$, and E_f is isogenous to a power of the Drinfeld module whose lattice is $\mathbb{F}_4[t, \sqrt[3]{t(t+1)}]$, the ring of integers in the fixed field $\mathsf{k}(\zeta_t \zeta_{t+1})$ of F. In this way we obtain the following result:

Corollary 6.2.1. *The t-module $E_{t(t-1)}$ is simple except in the following two cases:*

(a) *$E_{t(t-1)}$ is isogenous to H_3^2 when $q = 3$, where H_3 is the Drinfeld $\mathbb{F}_3[t]$-module with lattice $\mathbb{F}_3[\sqrt{-t}]$;*

(b) *$E_{t(t+1)}$ is isogenous to H_4^3 when $q = 4$, where H_4 is the Drinfeld $\mathbb{F}_4[t]$-module with lattice $\mathbb{F}_4[t, \sqrt[3]{t(t+1)}]$.*

We consider the exceptional cases $q = 3, 4$ in a bit more detail:
Case $q = 3$. From Proposition 4.5.3 we see that

$$\Gamma\left(\frac{1}{\theta(\theta-1)}\right) \sim \Gamma\left(\frac{\theta+1}{\theta(\theta-1)}\right).$$

Now we see that the sub-t-modules of E_f are isogenous to E_t. Moreover, taking into account (6.2) we see

$$\Gamma\left(\frac{1}{\theta}\right) \sim \Gamma\left(\frac{1}{\theta(\theta-1)}\right) \sim \Gamma\left(\frac{\theta+1}{\theta(\theta-1)}\right) \sim \widetilde{\pi}_\theta.$$

All of these equivalences are confirmed by the bracket relations. In fact, if we let Exp_t be the exponential function of E_t, then it can be shown that the exponential function Exp_f of E_f is

$$\mathrm{Exp}_f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \zeta_{\theta-1}\alpha \\ \beta & -\zeta_{\theta-1}\beta \end{pmatrix} \begin{pmatrix} \mathrm{Exp}_t(-z_1/\alpha - z_2/\beta) \\ \mathrm{Exp}_t(-z_1/\zeta_{\theta-1}\alpha + z_2/\zeta_{\theta-1}\beta) \end{pmatrix},$$

where $\Gamma(1/\theta(\theta-1)) = \alpha\theta(\theta-1)\Gamma(1/\theta)$ and $\Gamma((\theta+1)/\theta(\theta-1)) = \beta\Gamma(1/\theta)$ (cf. proof of Theorem 2.5.2).

Case $q = 4$. Here the monics in $\mathcal{I}_+$ are 1, $t+\ell$, $t+\ell^2$. The bracket relations as well as Proposition 4.5.3 confirm that

$$\Gamma\left(\frac{1}{\theta(\theta+1)}\right) \sim \Gamma\left(\frac{\theta+\ell}{\theta(\theta+1)}\right) \sim \Gamma\left(\frac{\theta+\ell^2}{\theta(\theta+1)}\right) \sim \widetilde{\pi}_{H_4},$$

where $\widetilde{\pi}$ is the period of H_4.

6.3. **Another Example.** $f = t(t-1)(t+1)$ and $q = 3$. In this example, the set of monic elements $\mathcal{I}_+$ of $(\mathsf{A}/f)^\times$ consists of four elements:

$$\mathcal{I}_+ = \{1, t^2+1, t^2+t-1, t^2-t-1\}.$$

We check that $\mathcal{I}_+$ is a subgroup of $(\mathsf{A}/f)^\times$ and that the fixed field of $\mathcal{I}_+$ is the field

$$\mathsf{L} := \mathsf{k}(\zeta_t\zeta_{t-1}\zeta_{t+1}) = \mathsf{k}(\sqrt{-(t^3-t)}).$$

The ring of integers in L forms a rank 2 lattice in C_∞ which corresponds to a Drinfeld $\mathbb{F}_3[t]$-module ψ with CM by the ring of integers of L. The t-module E_f is then isogenous to ψ^4. If we let $\widetilde{\pi}_\psi$ be a fundamental period of the Drinfeld module ψ, then

$$\Gamma\left(\frac{1}{\theta^3-\theta}\right) \sim \Gamma\left(\frac{\theta^2+1}{\theta^3-\theta}\right) \sim \Gamma\left(\frac{\theta^2+\theta-1}{\theta^3-\theta}\right) \sim \Gamma\left(\frac{\theta^2-\theta-1}{\theta^3-\theta}\right) \sim \widetilde{\pi}_\psi.$$

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Department of Mathematics, Penn State University, University Park, PA 16802
E-mail address: wdb@math.psu.edu

Department of Mathematics, Penn State University, University Park, PA 16802
E-mail address: map@math.psu.edu