On the Whitehead Determinant for Semi-local Rings

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We answer the question: when the Whitehead determinant of a semi-local ring is the abelization of the multiplicative group?

Introduction

On November 6, 2003, R. Sujatha <sujatha@math.tifr.res.in> asked me the following two questions:

(1) Let R be a (not necessarily commutative) semi-local ring. Is $K_1(R)$ isomorphic to $R^*/[R^*, R^*]$?

(2) Are there any 'special' (non-commutative) semi-local rings for which one could expect (1) to be true??

I referred her to [2], where Theorem 3.6 asserts that $K_1(R)$ is $R^*/\tilde{E}$, where $\tilde{E}$ is the group generated by $(1+xy)/(1+yx)$ with x, y in R and $1+xy$ in R^* and where the last sentence in §3 says that $\tilde{E}$ is not $[R^*, R^*]$ in the case when $R = M_2(Z/2Z)$ is the ring of 2 by 2 matrices over a field of two elements (Z is the ring of integers). Moreover, in this case the group $K_1(R)$ is trivial while $R/[R^*, R^*]$ has order two. Therefore $\tilde{E} \neq [R^*, R^*]$ whenever R has a ring morphism onto $M_2(Z/2Z)$, see Theorem 1 below.

Recall [1, p. 503] that a ring R is semi-local if and only if the ring $R/\mathrm{rad}(R)$ is isomorphic to a finite product of matrix rings over division rings D where $\mathrm{rad}(R)$ is the Jacobson radical of R. The Whitehead determinant $GL_n(R) \to K_1(R)$ was introduced for any associative ring R with 1 and any integer $n \geq 1$ in [1].

Here is another counter example to (1). Let $R = T_2(Z/2Z)$ be the ring of 2 by 2 upper triangular matrices over $Z/2Z$. In this case, $R/\mathrm{rad}(R)$ is isomorphic to $(Z/2Z) \times (Z/2Z)$, the multiplicative group $R^* = \tilde{E}$ has order two and its commutator subgroup is trivial.

Theorem 1. *Let R be an associative ring with 1 such that* $\mathrm{sr}(R) = 1$ *and R has a factor ring isomorphic to $M_2(Z/2Z)$ or $T_2(Z/2Z)$. Then the kernel of the Whitehead determinant $R^* \to K_1(R)$ is bigger than $[R^*, R^*]$.*

By[1], [2], R satisfies the first Bass stable range condition, which we write as $\mathrm{sr}(R) = 1$, if $R/\mathrm{rad}(R)$ is isomorphic to a product of full matrix rings over division rings D, e.g., R is semi-local.

So to answer the second question of Sujatha we must exclude factors in $R/\mathrm{rad}(R)$ which are isomorphic to $M_2(Z/2Z)$ (and hence have order 16), and we do not want more that one factor isomorphic to $Z/2Z$. We do not need the condition that the number of factors is finite.

THEOREM 2. *Let R be an associative ring with 1 such that $R/rad(R)$ is product of full matrix rings over division algebras. Assume that none of these matrix rings is isomorphic to $M_2(Z/2Z)$ and that no more than one of these matrix rings has order* 2. *Then $\tilde{E} = [R^*, R^*]$, hence $K_1(R) = R^*/[R^*, R^*]$.*

PROOF of THEOREM 1

Consider an isomorphism $R/J = R'$ where J is an ideal of R and the factor ring R' is isomorphic to $M_2(Z/2Z)$ or $T_2(Z/2Z)$. We set $x' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R'$. Then $(1 + x'y')(1 + y'x')^{-1} = 1 + x'y' \in R'^*$ has order 2 and belongs to the kernel of the Whitehead determinant, but does not belong to $[R'^*, R'^*]$. Recall that $[R'^*, R'^*]$ has order 3 in the case $R' = M_2(Z/2Z)$ and order 1 in the case $R' = T_2(Z/2Z)$.

Let $x, y \in R$ be the inverse images of x', y'. We set $z = (1 + xy)^2 - 1 \in J$. We have $R(1+xy)+Rz = R$ hence $R(1+xy)+Rzxy = R$. Since $\mathrm{sr}(R)$ =1, there is $r \in R$ such that $R(1+xy+azxy) = R$. By [3, Theorem 2.6], $1+xy+azxy \in R^*$. Set $x_0 = x+azx \in x+J$.

On one hand, the image of $(1+x_1y)(1+yx_1)^{-1}$ in K_1R is trivial. On the other hand, $(1 + x_0y)(1 + yx_0)^{-1}$ is not in $[R^*, R^*]$ because otherwise its image $1 + x'y'$ in R' would be in $[R'^*, R'^*]$.

REMARK. The condition $\mathrm{sr}(R) = 1$ in Theorem 1 is not redundant. For example, the free ring $R = Z < t_1, t_2 >$ has both $T_2(Z/2Z)$ and $M_2(Z/2Z)$ as factor rings. Namely,

$$t_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in T_2(Z/2Z), t_2 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in T_2(Z/2Z)$$

and

$$t_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(Z/2Z), t_2 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in T_2(Z/2Z).$$

On the other hand,

$$K_1(R) = K_1(Z) = \{\pm 1\} = Z^* = R^* = R^*/[R^*, R^*].$$

PROOF of THEOREM 2

We start our proof with two lemmas.

LEMMA 1. *Let D be a division algebra, $n \geq 1$ an integer, and $R = M_n(D)$ the ring of $n \times n$ matrices over D. When $n = 1$, assume that D has at least 3 elements. Then every element r in R is the sum of two units.*

Proof. When $n = 1$, we pick any $u \neq 0, -r$ and write $r = u + (r - u)$ with $u, r - u \in R^* = D^*$.

When $n \geq 2$, multiplying r on left and right by units (i.e., by invertible matrices) we can assume that all diagonal entries of r are zeros. Then we write r as sum of an upper triangular matrix with ones along the diagonal and an lower triangular matrix with negative ones along the diagonal. QED.

REMARK. 1 in $D = Z/2Z$ is not the sum of two units.

LEMMA 2. *Let D be a division algebra, $n \geq 1$ an integer, and $R = M_n(D)$. Then every $f \in R^* = GL_n(D)$ is a product uv with $u, v, u - 1, v - 1 \in R^*$ with the following three exceptions:*

(a) $\mathrm{card}(R) = 3$ *and* $f = -1$,

(b) $\mathrm{card}(R) = 2$,

(c) $\mathrm{card}(R) = 16$ *and f is order* 2 *(i.e.,* $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, *or* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. When D has at least 4 elements, we replace f by a similar matrix and assume that $f = u'dl$ where u' is an upper triangular matrix with ones on the diagonal, d is a diagonal matrix, and l is a lower triangular matrix with ones on the diagonal. Then we write $d = d'd''$ with diagonal matrices d', d'' without ones on the diagonal. Then we set $u = u'd', v = d''l \in R^*$. We have $f = uv$ with $u, v, u - 1_n, v - 1_n \in GL_n(D)$.

Assume now that D has only 3 elements and $n \geq 2$. Replacing f by a similar matrix, we can assume that f is an upper block triangular matrix, where each block is either a companion matrix of size $k \times k$ with $k \geq 2$, or 1_2, or 1_3, or -1_2, or -1_3.

We used that $\begin{pmatrix} \pm 1 & 0 \\ 0 & g \end{pmatrix}$ with a companion matrix g is similar to either companion matrix or $\begin{pmatrix} \pm 1_2 & * \\ 0 & h \end{pmatrix}$ with a companion matrix h (or h could be absent).

When $f = 1_k$, we have $f = uv$ with $u = v = -1_k$.

When $f = -1_k$ with $k = 2$ or 3, we have $f = uv$ with $u = -v^{-1}$ being the companion matrix of a polynomial not vanishing at both 1 and -1 (e.g., $p(x) = \lambda^2 + 1$ when $k = 2$ and $p(x) = \lambda^3 - \lambda + 1$ when $k = 3$.

Let now f be a companion matrix of size $k \times k$ with $k \geq 2$. There is an elementary matrix g such that fg is a companion matrix whose eigenvalues do not include -1. Then $f = uv$ with $u = -fg, v = -g^{-1}$.

Assume now that D has only 2 elements and $n \geq 2$. Our matrix f is similar to a direct sum of matrices each of them is either a companion matrix with 1 not an eigenvalue or an upper triangular matrix. Therefore it suffices to prove our conclusion in the following four cases:

(1) $f = 1_n$ (the identity matrix) with $n \geq 2$,

(2) all eigenvalues of f are 1 (i.e., the characteristic polynomial of f is $(\lambda - 1)^n$, i.e., f is similar to an upper triangular matrix) and $n \geq 3$,

(3) 1 is not an eigenvalue of f (i.e., $f - 1_n \in GL_n(D)$).

(4) f is a direct sum of a companion matrix with 1 not an eigenvalue and a $k \times k$ upper triangular matrix with $k \leq 2$.

In Case (1), $f = 1_n = uv = uu^{-1}$ where u is the companion matrix of the polynomial $\lambda^n + \lambda + 1$.

In Case (2), we can assume that f is an upper triangular matrix and that all entries of f outside the main diagonal and the line above are zeros. Also in view of Case (1) we can assume that $f_{12} = 1$. Let u be the companion matrix for the polynomial $\lambda^n + \lambda^{n-1} + 1$, i.e., f has ones at the line below the main diagonal and at the first and last positions at the last column while all other entries of f are zeros. Set $v = u^{-1} f$. The first row of v is $(0, \ldots, 0, 1)$, and if cross out the first row and the last column, we obtain an upper triangular matrix with ones along the main diagonal. So $v - 1 \in R^*$.

In Case (3), $f = uv$ with $u = f^2, v = f^{-1}$. Notice that $u - 1 = f^2 - 1 = (f-1)^2 \in R^*$ and $v - 1 = f^{-1} - 1 = f^{-1}(f - 1) \in R^*$.

In Case (4), f is similar to a companion matrix with an eigenvalue 1. So assume now that $n \geq 3$ and f is a companion matrix with an eigenvalue 1. We proceed by induction on n.

Let $n = 3$, $f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}$ with $a + b = 0$. We set $u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $v = u^{-1} f = \begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 1 \\ 1 & 0 & 1+a \end{pmatrix}$. Then $f = uv$ and $u, v, u-1, v-1 \in R^*$.

Let $n = 4$, $f = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix}$ with $a + b + c = 0$. We set $u = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $v =$ $v = u^{-1} f = \begin{pmatrix} 1 & 1 & 0 & a+b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1+a \end{pmatrix}$. Then $f = uv$ and $u, v, u-1, v-1 \in R^*$.

Let now $n \geq 5$ and f is a companion matrix. We set $g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus 1_{n-3} = g^{-1} \in GL_n(D)$. The matrix gfg^{-1} has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_2 & 0 \\ ca^{-1} & 1_{n-2} \end{pmatrix} (a \oplus (d - ba^{-1}c) \begin{pmatrix} 1_2 & a^{-1}b \\ 0 & 1_{n-2} \end{pmatrix}$$

with $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = a^{-2}$. By the induction hypothesis, we can write $d - ba^{-1}c) = u'v'$ with $u', v', u' - 1_{n-2}, v' - 1_{n-2} \in GL_{n-2}(D)$.

Then $gfg^{-1} = \begin{pmatrix} a^{-1} & 0 \\ * & u' \end{pmatrix} \begin{pmatrix} a^{-1} & * \\ 0 & v' \end{pmatrix}$, hence $g = uv$ with

$$u = g^{-1} \begin{pmatrix} a^{-1} & 0 \\ * & u' \end{pmatrix} g, v = g^{-1} \begin{pmatrix} a^{-1} & * \\ 0 & v' \end{pmatrix} g \in GL_n(D)$$

and $u - 1_n, v - 1_n \in GL_n(D)$. QED.

REMARK. All exceptions in Lemma 2 are necessary.

COROLLARY 1. Let $n \geq 1$ be an integer, D a division ring, $R = M_n(D)$, and $y \in R$. Then for any $x \in R$ such that $1 + xy \in R^*$ there are $x_1, x_2 \in R^*$ such that

$$x = x_1 + x_2 + x_1 y x_2$$

and $1 + x_1 y, 1 + x_2 y \in R^*$ with the following four exceptions:

(a) card(R)=3 *and* $xy \neq 0$,

(b) card(R)=2 *and either* $x = 1$ *or* $y = 1$,

(c) card(R)=16 *and* $y \in R^*$ *and* $1 + xy$ *has order 2 in* R^*,

(d) card(R)=16 *and the matrix* $y \in M_2(D)$ *has rank 1 and either* xy *or* yx *is nonzero.*

Proof. Note that for any associative ring R and any $y \in R$, the binary operation $(a, b) \mapsto a \circ b = a + b + ayb$ is a group operation on the set $\{a \in R : 1 + ay \in R*\}$. The neutral element is 0. The inverse of a is $-(1 + ay)^{-1}a = -a(1 + ya)^{-1}$. Note that this inverse belongs to R^* if and only if $a \in R^*$.

In the case when $y = 0$, this group is the additive subgroup of R. In the case when $y = 1$, the group is essentially R^*. Namely, $a + b + ab = (1 + a)(1 + b) - 1$. For an arbitrary y, we have $(1 + ay)(1 + by) = 1 + (a + b + ayb)y = 1 + (a \circ b)y$ and $(1 + ya)(1 + yb) = 1 + y(a + b + ayb) = 1 + y(a \circ b)$.

For any $u, v \in R^*$, we can multiply the equation $x = x_1 + x_2 + x_1 y x_2$ by u on the left and v on the right obtaining a similar equation with x, x_i, y replaced by $uxv, ux_i v, v^{-1} y u^{-1}$ respectively and preserving the conditions $1 + xy, 1 + x_i y \in R^*$.

In our special case $R = M_n(D)$, we can choose u, v such that $y = y^2$ is a diagonal matrix with k ones on the diagonal followed by $n - k$ zeros where $0 \leq k \leq n$.

When $k = 0$, our statement follows from Lemma 1. When $k = n$, our statement follows from Lemma 2.

Assume now that $1 \leq k \leq n - 1$ (so $n \geq 2$).

We write the given matrix x and unknown matrices x_i in block form:
$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ with $a, a_i \in M_k D$, etc.

Then
$$x_1 + x_2 + x_1 y x_2 = \begin{pmatrix} a_1 + a_2 + a_1 a_2 & b_1 + b_2 + a_1 b_2 \\ c_1 + c_2 + c_1 a_2 & d_1 + d_2 + c_1 b_2 \end{pmatrix}.$$

The condition $1_n + xy \in GL_n(D)$ means that $1 + a \in GL_k(D)$.

First we prove our corollary in the following case:

(1) $a * 1_k = uv$ with $u, v, u - 1_k, v - 1_k \in GL_k(D)$ and $d = d_1 + d_2$ with $d_i, d_i - 1_{n-k} \in GL_{n-k}(D)$.

In this case, our proof is easy: we set

$$x_1 = \begin{pmatrix} u - 1_k & b \\ 0 & d_1 \end{pmatrix}, x_2 = \begin{pmatrix} v - 1_k & 0 \\ c & d_2 \end{pmatrix} \in GL_n(D)$$

with $1_n + x_i y \in GL_n(D)$ and $x = x_1 + x_2 + x_1 y x_2$.

In some cases we will prove Corollary 1 using induction on n. We write $x = \begin{pmatrix} x' & b' \\ c' & d' \end{pmatrix}$ and $y = \begin{pmatrix} y' & 0 \\ 0 & 0 \end{pmatrix}$ wth $x', y' \in M_{n-1}(D), d' \in D$. Suppose that the following condition holds:

(2) $x' = x'_1 + x'_2 + x'_1 y' x'_2$ with $x'_i, 1_{n-1} + x'_i y' \in GL_{n-1}(D)$.

Then we prove our conclusion as follows.

If $d' = 0$, or $\mathrm{card}(D) \neq 2$, then we write $d' = d'_1 + d'_2$ with $d'_1, d'_2 \in D*$. Then we set $x_1 = \begin{pmatrix} x'_1 & b' \\ 0 & d'_1 \end{pmatrix}, x_2 = \begin{pmatrix} x'_1 & 0 \\ c' & d'_2 \end{pmatrix} \in GL_n(D)$ with $x = x_1 + x_2 + x_1 y x_2$ and $1 + x_i y \in GL_n(D)$.

Assume now that $d' \neq 0$ and $\mathrm{card}(D) = 2$ hence $d' = 1$.

If $n-k \geq 2$, then we can be reduced to the case $d' = 0$ replacing x and y by $x \begin{pmatrix} 1_k & 0 \\ 0 & g \end{pmatrix}$ and $y = \begin{pmatrix} 1_k & 0 \\ 0 & g \end{pmatrix}^{-1} y$ with a matrix $g \in GL_{n-k}(D)$.

Assume now that $d' = 1$, $\mathrm{card}(D) = 2$ and $n - k = 1$.

If $b' \neq 0$, we can be reduced to the case $d' = 0$ replacing x and y by $\begin{pmatrix} 1_{n-1} & 0 \\ * & 1 \end{pmatrix} x$ and $y = \begin{pmatrix} 1_{n-1} & 0 \\ * & 1 \end{pmatrix} y$

If $c' \neq 0$, we can be reduced to the case $d' = 0$ replacing x and y by $x \begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix} x$ and $y = y \begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix} y$.

Assume now that $\mathrm{card}(D) = 2, n - k = 1, d' = 1, b' = 0, c' = 0$.

If $x' \neq 0$, then $x_1'^{-1} + x_2'^{-1} + 1_{n-1} \neq 0$. So there are a column b and a row c such that $c(x_1'^{-1} + x_2'^{-1} + 1_{n-1})b = 1$. Now we set $u = x'_1 + 1_{n-1}, v = x'_2 + 1_{n-1} \in GL_{n-1}(D)$,

$$x_1 = \begin{pmatrix} x'_1 & ub \\ c & 1 + cx_1'^{-1}ub \end{pmatrix}, x_2 = \begin{pmatrix} x'_2 & b \\ cv & 1 + cvx_2'^{-1}b \end{pmatrix} \in GL_n(D).$$

Then $x = x_1 + x_2 + x_1 y x_2$ and $1 + x_i y \in GL_n(D)$.

Assume now that $\mathrm{card}(D) = 2, n - k = 1, d' = 1, b' = 0, c' = 0, x' = 0$.

If $n \geq 4$, let $u, u - 1_{n-2} \in GL_{n-2}(D)$. We set $x_1 = \begin{pmatrix} u + 1_{n-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} u^{-1} + 1_{n-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \in GL_n(D)$. Then $x = x_1 + x_2 + x_1 y x_2$ and $1 + x_i y \in GL_n(D)$.

Finally, assume that $\mathrm{card}(D) = 2, n - k = 1, d' = 1, b' = 0, c' = 0, x' = 0, n \leq 3$ Since $x'_1, x'_1 + 1_{n-1} \in GL_{n-1}(D)$, we conclude that $n \geq 3$. So $n = 3, k = 2$. We set $x_1 =$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in GL_3(D)$. Then $x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_1 + x_2 + x_1 y x_2$ and $1 + x_i y \in GL_3 D)$.

Thus, we have proved the corollary in Cases (1) and (2). In general, proceeding by induction on n when $n - k \geq 2$ and using Lemma 2 when $n - k = 1$, we are reduced to the following four cases (we write $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ again):

(3) $n = 2, k = 1$, card(D) =3, and $a = 1$;
(4) $n = 2, k = 1$, card(D) =2, and $a = b = c = 0$;
(5) $n = 3, k = 1$, and card$(D) = 2$;
(6) $n = 3, k = 2$, card$(D) = 2$, and $1 + a$ has order 2;

In Case (3), we set $x_1 = \begin{pmatrix} 0 & 1 \\ 1 & d - bc + b + c + 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 & b - 1 \\ c + 1 & bc + b - c \end{pmatrix} \in GL_2(D)$. Then $x = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = x_1 \circ x_2 = x_1 + x_2 + x_1 y x_2$.

In Case (4), we have $x = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = x_1 \circ x_2$ with $x_1 = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in GL_2(D), 1_2 + x_i y \in GL_2(D)$.

In Case (5), the condition $1 + ay \in GL_3(D)$ means $a = 0$.

Replacing x, x_i, y by $uxv, ux_iv, v^{-1}yu^{-1} = y$ respectively with $u, v \in GL_3(D)$ of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$, we can assume that:

$d_{2,2} = 0$ in the case when $b = 0, c = 0$;
$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the case when $b \neq 0$;
$c = (1, 1)$ in the case when $c \neq 0$.

When $b = 0$ and $c = 0$, we write $d = \begin{pmatrix} e_1 & e_2 \\ e_3 & 0 \end{pmatrix} \in M_2(D)$. Then

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_2 \\ 0 & e_3 & 0 \end{pmatrix} = x_1 \circ x_2 = x_1 + x_2 + x_1 y x_2$$

with

$$x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & e_1 & e_2 \\ 0 & e_3 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(D), 1_n + x_i y \in GL_3(D).$$

Here is how we handle the remaining cases:

$$x = \begin{pmatrix} 0 & 1 & 1 \\ 0 & e_1 & e_2 \\ 0 & e_3 & e_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & e_1 & e_2 \\ 0 & e_3 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & e_4 + 1 \end{pmatrix};$$

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 1 & e_1 & e_2 \\ 1 & e_3 & e_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & e_1 & e_2+1 \\ 0 & e_3 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & e_4+1 \end{pmatrix};$$

$$x = \begin{pmatrix} 0 & 1 & 1 \\ 1 & e_1 & e_2 \\ 1 & e_3 & e_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & e_1 & 1 \\ 1 & e_3+1 & e_4 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & e_2+1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In Case (6), replacing x, x_i, y by $uxu^{-1}, ux_iu^{-1}, uyu^{-1} = y$ respectively with $u \in GL_3(D)$ of the form $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$, we can assume that $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Replacing x, x_i, y by $uxv, ux_iv, v^{-1}yu^{-1} = y$ respectively with $u = \begin{pmatrix} 1_2 & 0 \\ * & 1 \end{pmatrix}, v = \begin{pmatrix} 1_2 & * \\ 0 & 1 \end{pmatrix} \in GL_3(D)$ (which does not change a), we can assume that $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e_1 \\ e_2 & 0 & d \end{pmatrix}$.

If $e_2 = 0$, we set $x_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & e_1 \\ 0 & 1 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1+d \end{pmatrix} \in GL_3(D)$.

If $e_1 = 0$, we set $x_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ e_2 & 1 & d \end{pmatrix} \in GL_3(D)$.

If $e_1 = e_2 = 1$, we set $x_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & d+1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in GL_3(D)$. QED.

Remark. All exceptions in Corollary 1 are necessary.

Proposition 1. Let R be an associative ring with 1 such that $R/\mathrm{rad}(R)$ is product of full matrix rings over division rings. Let $y \in R$ and assume that y belongs to every ideal of index 2,3 or 16 in R. Let $x \in R$ by such that $1 + xy \in R^*$ and that x belongs to any ideal of index two in R. Then there are $x_1, x_2 \in R*$ such that $1 + x_iy \in R^*$ for $i = 1, 2$ and $x = x_1 \circ x_2 = x_1 + x_2 + x_1yx_2$.

Proof. Using Corollary 1, we can find the components x_1', x_2' of all x_1, x_2 in every matrix ring R' with card$(R') \neq 2, 3$, or 16. In the case of card(R')=2, we set $x_1' = x_2' = 1$. In the case of card(R')=3, we set $x_1' = x_2' = -x'$. In the case of card(R')=16, we use Lemma 1.

Then x_1, x_2 are defined modulo rad(R), $x_1, x_2, 1 + x_1y, 1 + x_2y \in R^*$, and $x \equiv x_1 \circ x_2$ modulo rad(R). Now we replace x_2 by $(-(1+x_1y)^{-1}x_1) \circ x \in R^*$ and obtain that $1+x_2y \in R^*$ and $x = x_1 \circ x_2$. (Recall that $(-(1 + x_1y)^{-1}x_1) \circ x_1 = 0$.) QED.

Corollary 2. Under the conditions of Proposition 1, $(1+xy)(1+yx)^{-1}$ is a product of two commutators.

Proof. We have

$1 + xy = (1 + x_1y)(1 + x_2y)$

and

$1 + yx = (1 + yx_1)(1 + yx_2)$

with $x_1, x_2 \in R^*$. So

$$(1+x_iy)(1+yx_i)^{-1} = (1+x_iy)x_i^{-1}(1+x_iy)^{-1}x_i = [1+x_iy, x_i^{-1}]$$

is a commutator for $i = 1, 2$, hence

$$(1+xy)(1+yx)^{-1} = (1+x_1y)(1+x_2y)(1+yx_2)^{-1}(1+yx_1)^{-1}$$

$$= (1+x_1y)[1+x_2y, x_2^{-1}](1+yx_1)^{-1}$$

$$= [1+x_1y, x_1^{-1}][(1+yx_1)(1+x_2y)(1+yx_1)^{-1}, (1+yx_1)x_2^{-1}(1+yx_1)^{-1}]$$

is a product of two commutators.

Corollary 3. Let R be an associative ring with 1 such that $R/\mathrm{rad}(R)$ is product of full matrix rings over division rings. Let $x, y \in R$ and $1+xy \in R^*$. Assume that y belongs to every ideal of index 16 in R and that both x and y belong to every ideal of index 2 in R. Then $(1+xy)(1+yx)^{-1}$ is a product of four commutators.

Proof. If R has no ideals of index 3 or y belongs to all such ideals J of index 3 in R, then $(1+xy)(1+yx)^{-1}$ is a product of two commutators by Corollary 2.

Otherwise, we find $x_1, x_2 \in R^*$ such that $1+x_iy \in R^*$ and $x' = x_1' \circ x_2'$ in every factor matrix ring R' except for R' such that $\mathrm{card}(R') = 3$ and $x'y' \neq 0$. (If $\mathrm{card}(R') = 16$, we use Lemma 1.) In the exceptional case, $x_1' = x_2' = x' = y' = \pm 1$ and $x_1' \circ x_2' = 0$.

We set $\tilde{x} = x_1 \circ x_2 \circ x$. Then $1+\tilde{x}^2 \in R^*$. We set $\tilde{y} = y + \tilde{x} + y + y\tilde{x}^2 \in R$ with $1+\tilde{x}\tilde{y} \in R^*$. Then $\tilde{y}$ belongs to every ideal J of index 2 or 3 in R. By Corollary 2,

$$(1+\tilde{x}\tilde{y})(1+\tilde{y}\tilde{x})^{-1} = (1+\tilde{x}y)(1+\tilde{x}^2)(1+\tilde{x}^2)^{-1}(1+y\tilde{x})^{-1}$$

$$= (1+\tilde{x}y)(1+y\tilde{x})^{-1} = (1+x_1y)(1+x_2y)(1+xy)((1+yx_1)(1+yx_2)(1+yx))^{-1}$$

is a product of two commutators, hence $(1+xy)(1+yx)^{-1}$ is a product of four commutators. QED.

Proposition 2. Under the conditions of Theorem 2, let $x, y \in R$ and $1+xy \in R^*$. Then $(1+xy)(1+yx)^{-1}$ is a product of five commutators.

Proof. If R has no ideals of index 2 or there is (exactly one) such an ideal J_2 and $x, y \in J_2$, then $(1+xy)(1+yx)^{-1}$ is a product of four commutators by Corollary 3.

If $y \notin J_2$, then $x \in J_2$ (since $1+xy \in R^*$). By Lemma 1, there is $y_0 \in R^*$ such that $1+xy_0 \in R^*$ (i.e., $x+y_0^{-1} \in R^*$. Set $\tilde{y} = y + y_0 + yxx_0$. By Corollary 3,

$$(1+x\tilde{y})(1+\tilde{y}x)^{-1} = (1+xy)(1+xy_0)(1+y_0x)^{-1}(1+yx)^{-1}$$

is a product of four commutators, hence $(1+xy)(1+yx)^{-1}$ is a product of five commutators (because $(1+xy_0)(1+y_0x)^{-1}$ is a commutator).

If $x \notin J_2$, then $y \in J_2$. By Lemma 1, there is $x_0 \in R^*$ such that $1+x_0y \in R^*$. Set $\tilde{x} = x + x_0 + xyx_0$. By Corollary 3,

$$(1+\tilde{x}y)(1+y\tilde{x})^{-1} = ((1+y\tilde{x})(1+\tilde{x}y)^{-1})^{-1} = (1+xy)(1+x_0y)(1+yx_0)^{-1}(1+yx)^{-1}$$

is a product of four commutators, hence $(1+xy)(1+yx)^{-1}$ is a product of five commutators. QED.

Now we can finish our proof of Theorem 2. If R be an associative ring with 1 such that $R/\mathrm{rad}(R)$ is product of full matrix rings over division rings, then by [2, Theorem 3.6], the kernel of the Whitehead determinant $R^* = GL_1(R) \to K_1(R)$ is the subgroup $\tilde{E}$ of R^* generated by all $(1+xy)(1+yx)^{-1}$ with $x, y \in R$ and $1+xy \in R^*$. So Theorem 2 follows from Proposition 2.

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